Relativistic corrections in the Faddeev-Yakubovsky framework

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Motivation

Dynamical equations and interactions

Matrix elements of kinetic energy & free propagator

Numerical solutions of Faddeev- and Yakubovsky eq

Comparison of relativistic and non-relativistic solutions

Conclusions and Outlook

in collaboration with Arseniy Filin, Vadim Baru, Evgeny Epelbaum, Christopher Körber, Hermann Krebs, Daniel Möller, Patrick Reinert

Motivation

- high precision chiral NN interactions
  upcoming higher order 3N interactions
  → relativistic effects will certainly become more relevant

- high momentum observables are of interest:
  e.g form factors around minimum, pion capture, …

- SCS, SMS interaction allow to consider high momentum tail of bound state wave functions → importance of possible rel. effects ?

- current: high precision radius calculation for ^{3}He, ^{3}H and ^{4}He (see Arseniy!)

Can we estimate the size of relativistic effects?

Are relativistic effects relevant for low energy observables like radii?

Basic Idea

Employ two-body interaction adjusted to rel. kinetic energy

usual approach: start with an NN interaction define for a non-relativistic Schrödinger eq.

$$\left[\sum_i \frac{\vec{k}_i^{\,2}}{2m_i} + \sum_{i<j} V_{ij} + \sum_{i<j<k} V_{ijk}\right] \left|\Psi\right\rangle = E \left|\Psi\right\rangle \quad \text{with} \quad \sum_i \vec{k}_i \stackrel{!}{=} 0$$

Employ transformation to redefine NN interaction for a relativistic kinetic energy

(see Kamada & Glöckle PLB 655, 119 (2007))

Two-body rel. Schrödinger equation:

$$\left[2\underbrace{\sqrt{m^2 + \vec{k}^{\,2}}}_{\equiv \omega(k)} + v_{12}\right] \left|\Psi\right\rangle = 2\sqrt{m^2 + k_0^2} \left|\Psi\right\rangle$$

where $V_{ij} = \frac{1}{4m}\left(2\omega(k)v_{ij} + 2v_{ij}\omega(k) + v_{ij}^2\right)$

This relation relates the non-rel. and rel. interaction (to be solved iteratively).

Interactions are significantly different.

Wave functions in the **two-body system** are the same.

Transformation performed by Evgeny.

Omitted boost corrections

In the $A > 2$ systems, the NN subsystems are not in their rest frames.

Define the NN interaction as difference of interacting and non-interacting

$$v_{ij}(P) \equiv \sqrt{\left[2\omega(\vec{k}) + v_{ij}\right]^2 + \vec{P}_{ij}^{\,2}} - \underbrace{\sqrt{4\omega(\vec{k})^2 + \vec{P}_{ij}^{\,2}}}_{\equiv 2\omega_P(\vec{k})}$$

Defines the boosted interaction by (to be solved iteratively as before)

$$4mV_{ij} - v_{ij}^2 = 2\left(\omega(k)v_{ij} + v_{ij}\omega(k)\right)$$

Exact solution possible, but **not done here!**

Approximate solution (**also not done here!**): $v_{ij}(P)(\vec{k}, \vec{k}') = \left(1 - \frac{\vec{P}^2}{8\omega\left(\vec{k}\right)\omega\left(\vec{k}'\right)}\right)$

Assumption here: the boost can be approximated by 3NF, 4NF, … terms.

Few-nucleon Schrödinger equation

Non-relativistic equation

$$\left[\sum_i \frac{\vec{k}_i^{\,2}}{2m_i} + \sum_{i<j} V_{ij} + \sum_{i<j<k} V_{ijk}\right] \left|\Psi\right\rangle = E \left|\Psi\right\rangle$$

$$\sum_i \vec{k}_i \stackrel{!}{=} 0$$

"Relativization" leads to

$$\left[\sum_i \left(\sqrt{\vec{k}_i^{\,2} + m_i^2} - m_i\right) + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk}\right] \left|\Psi\right\rangle = E \left|\Psi\right\rangle$$

v_{ij} exact phase shift equivalent to chiral interaction

V_{ijk} chiral 3NF, parameters adjusted to relativistic calculation (problem: nd data not used)

→ same Jacobi coordinates can be used since they define the CM coordinates

$$A = 3: \quad \vec{p}_{ij} = \frac{1}{m_i + m_j}\left(m_j \vec{k}_i - m_i \vec{k}_j\right) \qquad \vec{p}_k = \frac{1}{m_i + m_j + m_k}\left((m_i + m_j)\vec{k}_k - m_k\left(\vec{k}_i + \vec{k}_j\right)\right)$$

$$A = 4: \quad \vec{p}_{kl} = \frac{1}{m_k + m_l}\left(m_l \vec{k}_k - m_k \vec{k}_l\right) \qquad \vec{q}_l = \frac{1}{m_i + m_j + m_k + m_l}\left((m_i + m_j + m_k)\vec{k}_l - m_l\left(\vec{k}_i + \vec{k}_j + \vec{k}_k\right)\right)$$

$$\vec{q}_{ij,kl} = \frac{1}{m_i + m_j + m_k + m_l}\left((m_k + m_l)\left(\vec{k}_i + \vec{k}_j\right) - (m_i + m_j)\left(\vec{k}_k + \vec{k}_l\right)\right)$$

Faddeev & Yakubovsky equation

Rewrite for bound state problem ($A = 3$)

$$\psi_{12} = G_0 t_{12} P \psi_{12} + \left(1 + G_0 t_{12}\right) G_0 V^{(3)}_{123} (1 + P) \psi_{12} \quad \text{where} \quad P = P_{12} P_{23} + P_{13} P_{23}$$
$$\Psi = (1 + P)\, \psi_{12}$$

Similarly Yakubovsky equations ($A = 4$)

$$\psi_1 = G_0 t_{12} P \left[\left(1 - P_{34}\right) \psi_1 + \psi_2\right] + \left(1 + G_0 t_{12}\right) V^{(3)}_{123} \Psi$$
$$\psi_2 = G_0 t_{12} \tilde{P} \left[\left(1 - P_{34}\right) \psi_1 + \psi_2\right]$$

where $\Psi = \left[1 - (1 + P) P_{34}\right](1 + P)\psi_1 + (1 + P)\left(1 + \tilde{P}\right)\psi_2$

and $\tilde{P} = P_{13} P_{24}$

- Relativization affects the free propagator and also t.
- **Permutations and coordinate transformations are not affected!**
- Formally, the t matrix is obtained from the same Lippmann-Schwinger equation

$$t_{12} = v_{12} + v_{12} G_0 t_{12}$$

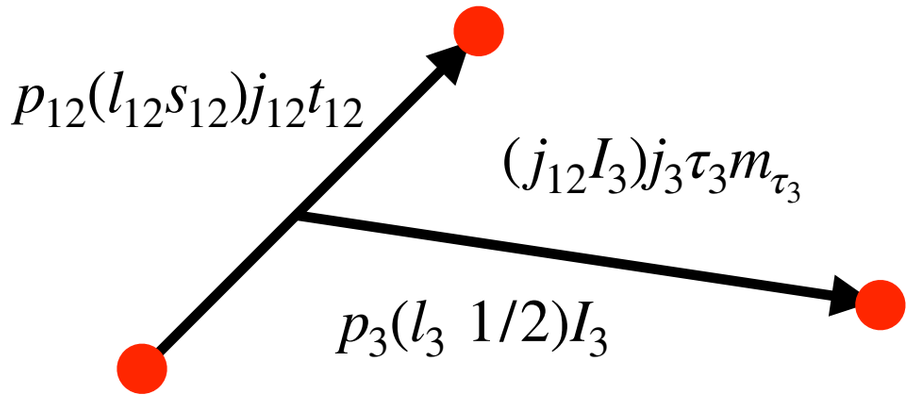
Both Yakubovsky components are expanded in different sets of Jacobi momenta!

Free propagator

Non-relativistic free propagator is the most simple part of the calculations, i.e.

$$\left\langle p'_{12} p'_3 \alpha' \right| G_0 \left| p_{12} p_3 \alpha \right\rangle = \delta_{\alpha'\alpha} \frac{\delta\left(p_{12} - p'_{12}\right)}{p_{12} p'_{12}} \frac{\delta\left(p_3 - p'_3\right)}{p_3 p'_3} \frac{1}{E - \frac{p_{12}^2}{m} - \frac{3p_3^2}{4m}}$$

Relativization induces angular dependence on angle between $\vec{p}_{12}$ and $\vec{p}_3$



in 3D:

$$\left\langle \vec{p}'_{12} \vec{p}'_3 \right| G_0 \left| \vec{p}_{12} \vec{p}_3 \right\rangle = \frac{\delta\left(\vec{p}_{12} - \vec{p}'_{12}\right) \delta\left(\vec{p}_3 - \vec{p}'_3\right)}{E - \sqrt{\left(\vec{p}_{12} - \frac{1}{2}\vec{p}_3\right)^2 + m^2} - \sqrt{\left(-\vec{p}_{12} - \frac{1}{2}\vec{p}_3\right)^2 + m^2} - \sqrt{p_3^2 + m^2} + 3m}$$

- Spin/isospin parts is conserved (assuming now equal masses).
- Expression uses that 3N system is in CM: $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$
- Rotational invariance, total angular momentum conservation.

Free propagator

Partial wave decomposition

$$G_0\left(p_{12}p_3x\right) = \frac{1}{E - \sqrt{p_{12} + \frac{1}{4}p_3^2 - p_{12}p_3x + m^2} - \sqrt{p_{12} + \frac{1}{4}p_3^2 + p_{12}p_3x + m^2} - \sqrt{p_3^2 + m^2} + 3m}$$

$$x = \widehat{p_{12}} \cdot \widehat{p_3}$$

Using

$$g_k\left(p_{12}p_3\right) = \int_{-1}^{1} dx\ P_k(x)\ G_0\left(p_{12}p_3x\right)$$ (due to symmetry $x \leftrightarrow -x$ only even k contribute)

(same angular dependence as kinetic energy)

$$\left\langle p'_{12}p'_3\alpha' \right| G_0 \left| p_{12}p_3\alpha \right\rangle = \frac{\delta\left(p_{12} - p'_{12}\right)}{p_{12}p'_{12}} \frac{\delta\left(p_3 - p'_3\right)}{p_3p'_3} \delta_{s_{12}s'_{12}}\delta_{j_3j'_3}\delta_{t_{12}t'_{12}}\delta_{\tau_3\tau'_3} \frac{1}{2} \sum_k \sum_{LS} \sqrt{\hat{j}_{12}\hat{I}_3\hat{j}'_{12}\hat{I}'_3}\ \hat{L}\hat{S}$$

$$\begin{Bmatrix} l_{12} & s_{12} & j_{12} \\ l_3 & 1/2 & I_3 \\ L & S & j_3 \end{Bmatrix} \begin{Bmatrix} l'_{12} & s_{12} & j'_{12} \\ l'_3 & 1/2 & I'_3 \\ L & S & j_3 \end{Bmatrix} \hat{k}^{3/2}(-1)^k \sqrt{\hat{l}_{12}\hat{l}_3\hat{L}} \begin{Bmatrix} k & k & 0 \\ l_{12} & l_3 & L \\ l'_{12} & l'_3 & L \end{Bmatrix}$$

$$\left(kl_{12}l'_{12},000\right)\left(kl_3l'_3,000\right) g_k(p_{12}p_3)$$

parity of individual subsystem is conserved, spin and isospin (for equal masses)

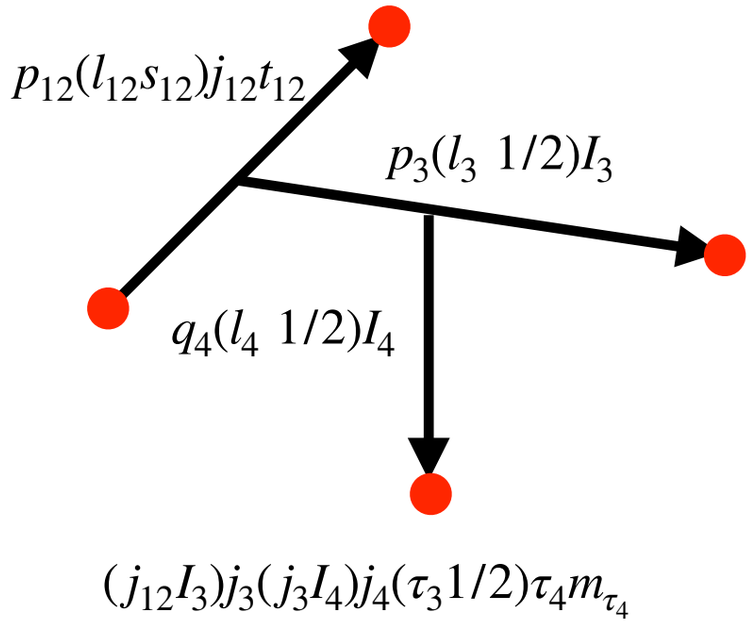
Free propagator $A = 4$

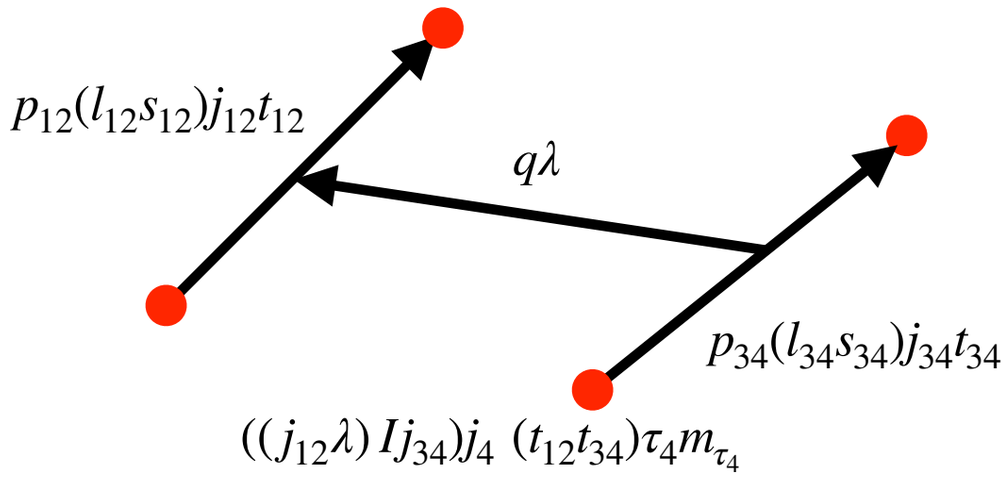
Complications: additional angular dependences & 2 kinds of coordinates

$$G_0\left(\vec{k_1}\vec{k_2}\vec{k_3}\vec{k_4}\right) = \frac{1}{E - \sqrt{k_1^2+m^2} - \sqrt{k_2^2+m^2} - \sqrt{k_3^2+m^2} - \sqrt{k_4^2+m^2} + 4m}$$

$$\vec{k_1} = +\vec{p_{12}} - \frac{1}{2}\vec{p_3} - \frac{1}{3}\vec{q_4} = \vec{p_{12}} + \frac{1}{2}\vec{q} \qquad \vec{k_3} = \vec{p_3} - \frac{1}{3}\vec{q_4} = \vec{p_{34}} - \frac{1}{2}\vec{q}$$

$$\vec{k_2} = -\vec{p_{12}} - \frac{1}{2}\vec{p_3} - \frac{1}{3}\vec{q_4} = -\vec{p_{12}} + \frac{1}{2}\vec{q} \qquad \vec{k_4} = \vec{q_4} = -\vec{p_{34}} - \frac{1}{2}\vec{q}$$





3+1 coordinates: three relative angles, in first step avoid implementation

2+2 coordinates: only two relevant angles, in first step application done in these coordinates

Partial wave decomposition in 2+2

Angular dependence on two angles, expansion in Legendre polynomials:

$$g_{k_{12}k_{34}}\left(p_{12}p_{34}q\right) = \int_{-1}^{1} dx_{12} \int_{-1}^{1} dx_{34}\; P_{k_{12}}(x_{12})\; P_{k_{34}}(x_{34})\; G_0\left(p_{12}p_{34}q, x_{12}x_{34}\right)$$

(due to symmetry $x \leftrightarrow -x$ only even k contribute)

$$\left\langle p'_{12}p'_{34}q'\alpha' \right|\; G_0 \;\left| p_{12}p_{34}q\alpha \right\rangle = \frac{\delta\left(p_{12}-p'_{12}\right)}{p_{12}p'_{12}} \frac{\delta\left(p_{34}-p'_{34}\right)}{p_{34}p'_{34}} \frac{\delta\left(q-q'\right)}{qq'} \delta_{s_{12}s'_{12}} \delta_{s_{34}s'_{34}} \delta_{j_4j'_4} \delta_{t_{12}t'_{12}} \delta_{t_{34}t'_{34}} \delta_{\tau_4\tau'_4}$$

$$\frac{1}{4} \sum_{K'L'} \sum_{KLS\mathscr{L}} (-1)^{I+j_{34}+I'+j'_{34}} \sqrt{\hat{j}_{12}\hat{j}_{34}\hat{I}\hat{j}'_{12}\hat{j}'_{34}\hat{I}'\hat{\lambda}\hat{l}_{12}\hat{l}_{34}}\; \hat{K}\hat{K}'\hat{L}\hat{L}'\hat{S}(\hat{\Lambda}\hat{\mathscr{L}})^{3/2}$$

$$\begin{Bmatrix} l_{12} & s_{12} & j_{12} \\ l_{34} & s_{34} & j_{34} \\ L & S & K \end{Bmatrix} \begin{Bmatrix} l'_{12} & s_{12} & j'_{12} \\ l'_{34} & s'_{34} & j'_{34} \\ L' & S & K' \end{Bmatrix} \begin{Bmatrix} j_{34} & j_{12} & K \\ \lambda & j_4 & I \end{Bmatrix} \begin{Bmatrix} j'_{34} & j'_{12} & K' \\ \lambda' & j_4 & I' \end{Bmatrix} \begin{Bmatrix} \lambda & L & \mathscr{L} \\ S & j_4 & K \end{Bmatrix} \begin{Bmatrix} \lambda' & L' & \mathscr{L} \\ S & j_4 & K' \end{Bmatrix}$$

$$\sum_{k_{12}k_{34}} \left(\hat{k}_{12}\hat{k}_{34}\right)^{3/2} \left(k_{12}l_{12}l'_{12},000\right) \left(k_{34}l_{34}l'_{34},000\right) \left(k_{12}k_{34}\Lambda,000\right) \left(\Lambda\lambda\lambda',000\right)$$

$$\begin{Bmatrix} k_{12} & k_{34} & 0 \\ k_{12} & k_{34} & 0 \\ \Lambda & \Lambda & 0 \end{Bmatrix} \begin{Bmatrix} k_{12} & k_{34} & \Lambda \\ l_{12} & l_{34} & L \\ l'_{12} & l'_{34} & L' \end{Bmatrix} \begin{Bmatrix} \Lambda & \Lambda & 0 \\ L & \lambda & \mathscr{L} \\ L' & \lambda' & \mathscr{L} \end{Bmatrix} g_{k_{12}k_{34}}(p_{12}p_{34}q)$$

- partial wave couplings spread over complete 4N space
- conversion of spin, subsystem parity will not be used in the following

Lippmann-Schwinger equation

formulation of Faddeev- Yakubovsky equations requires *t*-matrices

only kinetic energy is relativized (different from Kadyshevsky eq)

similar equation for 3N!

$$t_{\alpha'\alpha}(p'_{12},p_{12};p_{34}q) = v_{\alpha'\alpha}(p'_{12},p_{12}) + \sum_{\alpha''\alpha'''}\int dp''_{12}{p''_{12}}^2\; v_{\alpha'\alpha''}(p'_{12},p''_{12})\; G_0^{\alpha''\alpha'''}(p''_{12}p_{34}q)\; t_{\alpha'''\alpha}(p''_{12},p_{12};p_{34}q)$$

where $G_0^{\alpha''\alpha'''}(p''_{12}p_{34}q) = \sum_{k_{12}k_{34}} C^{k_{12}k_{34}}_{\alpha''\alpha'''}\; g_{k_{12}k_{34}}(p''_{12}p_{34}q)$

application to 4N amplitude in 2+2 coordinate

$$\left\langle p'_{12}p'_{34}q'\alpha'\right|\; t_{12}\;\left|\psi\right\rangle = \sum_{\alpha}\int dp_{12}p_{12}^2\; t_{\alpha'\alpha}(p'_{12},p_{12};p_{34}q)\;\psi_\alpha\left(p_{12}p_{34}q\right)$$

application to 4N amplitude in 3+1 coordinates (summation over intermediate states)

$$\left\langle p'_{12}p'_3q'_4\alpha'\right|\; t_{12}\;\left|\psi\right\rangle = \overbrace{\left\langle p'''_{12}p'''_3q'''_4\alpha'''\right|\; p''_{12}p''_{34}q''\alpha''\right\rangle}^{\text{transformation}}\;\overbrace{\left\langle p''_{12}p''_{34}q''\alpha''\right|\; t_{12}\;\left|p'_{12}p'_{34}q'\alpha\right\rangle}^{t\text{-matrix application}}$$

$$\underbrace{\left\langle p'_{12}p'_{34}q'\alpha'\right|\; p_{12}p_3q_4\alpha\right\rangle}_{\text{transformation}}\left\langle p_{12}p_3q_4\alpha\;\right|\psi\rangle$$

effectively, ψ_1 is projected on 2+2 coordinates (inefficient!)

Lippmann-Schwinger equation

quick & dirty coding (expecting only few calculations!)

- Lippmann-Schwinger equation is high dimensional
  preparation of t-matrix is major part of calculation
- solution depends on p_{34} and q and partial wave individually

  $\longleftrightarrow$ non-rel.: interpolation from grid of off-shell energies

- conservation of spin, parity, isospin not used yet
- iterative solution could be implemented
- direct implementation of t-matrix in 3+1 coordinates (when solved iteratively)
  likely improved partial wave convergence

- interaction not boosted

$$t_{\alpha'\alpha}(p'_{12}, p_{12}; p_{34}q) = v_{\alpha'\alpha}(p'_{12}, p_{12}) + \sum_{\alpha''\alpha'''} \int dp''_{12} {p''_{12}}^2 \; v_{\alpha'\alpha''}(p'_{12}, p''_{12}) \; G_0^{\alpha''\alpha'''}(p''_{12} p_{34} q) \; t_{\alpha'''\alpha}(p''_{12}, p_{12}; p_{34}q)$$

Comparison in 3N

Comparison of non-relativistic and rel. calculations

NN interaction completely phase equivalent

3N interaction requires adjustment of parameters: c_D , c_E and c_{E1}

but first test: **without 3NF** (potnr=92 = rel. interaction, potnr=90 non. rel. interaction)

potnr	cd	ce	cE1	mt3	E[MeV]	H[MeV]	T[MeV]	r(p)[fm]
90.0	0.0	0.0	0.0	-1.0	-8.077650	-8.07856	34.4113	1.60410
92.0	0.0	0.0	0.0	-1.0	-8.106181	-8.10670	33.7827	1.59997
90.0	0.0	0.0	0.0	1.0	-7.342120	-7.34320	33.6506	1.80813
92.0	0.0	0.0	0.0	1.0	-7.368389	-7.37018	33.0306	1.80217

- change of results not dramatic, but visible
- relativistic calculation is more bound

no 3NF = no boost corrections at all

Comparison in 3N

with 3NF (including relevant boost corrections?)

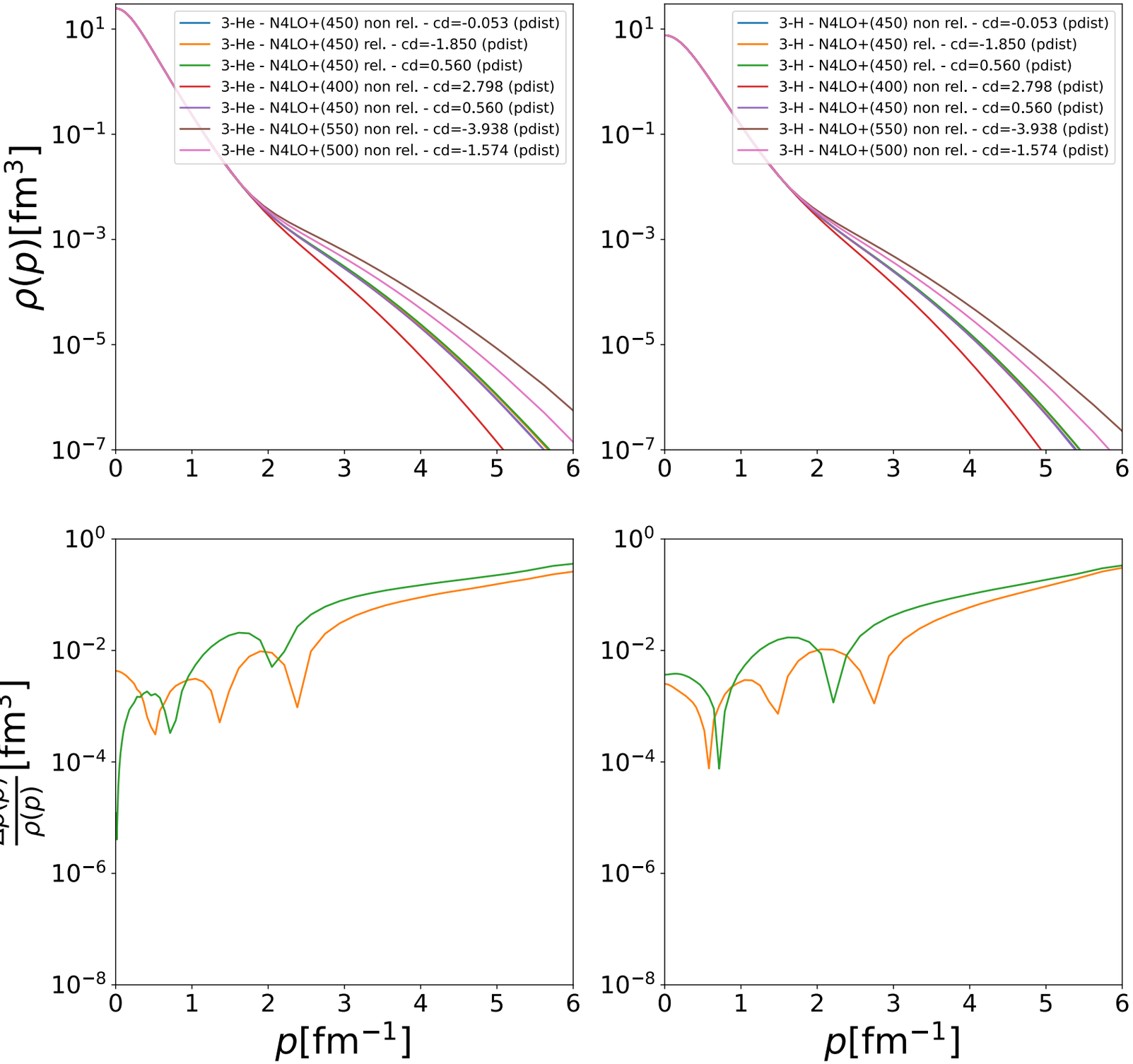
potnr	cd	ce	cE1	mt3	E[MeV]	H[MeV]	T[MeV]	r(p)[fm]
90.0	0.55973	-0.31394	0.0	-1.0	-8.460717	-8.46100	35.7586	1.58025
92.0	0.55973	-0.34498	0.0	-1.0	-8.458341	-8.45834	35.0276	1.57953
90.0	0.55973	-0.31394	0.0	1.0	-7.720559	-7.72102	35.0217	1.77172
92.0	0.55973	-0.34498	0.0	1.0	-7.718420	-7.71842	34.3052	1.76973

- binding energies are adjusted (small variation of energy is less important)
- radius change of 0.002 fm is significant
- significant change of kinetic energy (and potential energy) contribution

→ there is a possibility of additional uncertainty in radius calculations (see Arseniy!)

Motivates new fits for relativistic ^{4}He

Comparison in 3N



change of momentum distribution is very small.

cutoff variation of non-relativistic results more visible, but less relevant for radius

Comparison in 3N

cutoff dependence — binding energies

potnr	LambdaNN	cd	ce	cE1	mt3	E[MeV]	H[MeV]	T[MeV]	r(p)[fm]
90.0	400.0	2.79827	-0.38452	0.0	-1.0	-8.467830	-8.46864	33.6775	1.57766
90.0	450.0	0.55973	-0.31394	0.0	-1.0	-8.460626	-8.46148	35.7584	1.58026
90.0	500.0	-1.57418	-0.28583	0.0	-1.0	-8.454470	-8.45534	38.4927	1.58235
90.0	550.0	-3.93824	-0.25317	0.0	-1.0	-8.447442	-8.44830	41.6099	1.58412
90.0	400.0	2.79827	-0.38452	0.0	1.0	-7.719499	-7.72047	32.9399	1.76912
90.0	450.0	0.55973	-0.31394	0.0	1.0	-7.720343	-7.72135	35.0211	1.77175
90.0	500.0	-1.57418	-0.28583	0.0	1.0	-7.720834	-7.72185	37.7362	1.77399
90.0	550.0	-3.93824	-0.25317	0.0	1.0	-7.720896	-7.72190	40.8367	1.77586

binding energies are fitted

radius dependence will be compensated by two-nucleon currents (see Arseniy!)

Extension to 4N

start without adaption of binding energy using c_{E1}

potnr	cd	ce	cE1	E[MeV]	H1[MeV]	H2[MeV]	T[MeV]	r(p)[fm]
90.0	0.559730	-0.31394	0.0	-28.213733	-28.2104	-28.2010	73.9496	1.43490
92.0	0.559731	-0.34498	0.0	-27.989544	-28.0033	-28.0078	71.8733	1.43056

3N binding energies and c_D are fixed

small but visible shift of radius found (consistent with change in 3N)

radius will decrease with increasing energy!

binding energy change can be estimated assuming that binding momentum stays similar (like in NN)

$$B_r - B_{nr} = \sqrt{3m + p_0^2} - 3m + \sqrt{m + p_0^2} - m - \frac{2p_0^2}{3m} \approx -240\,\mathrm{keV}$$

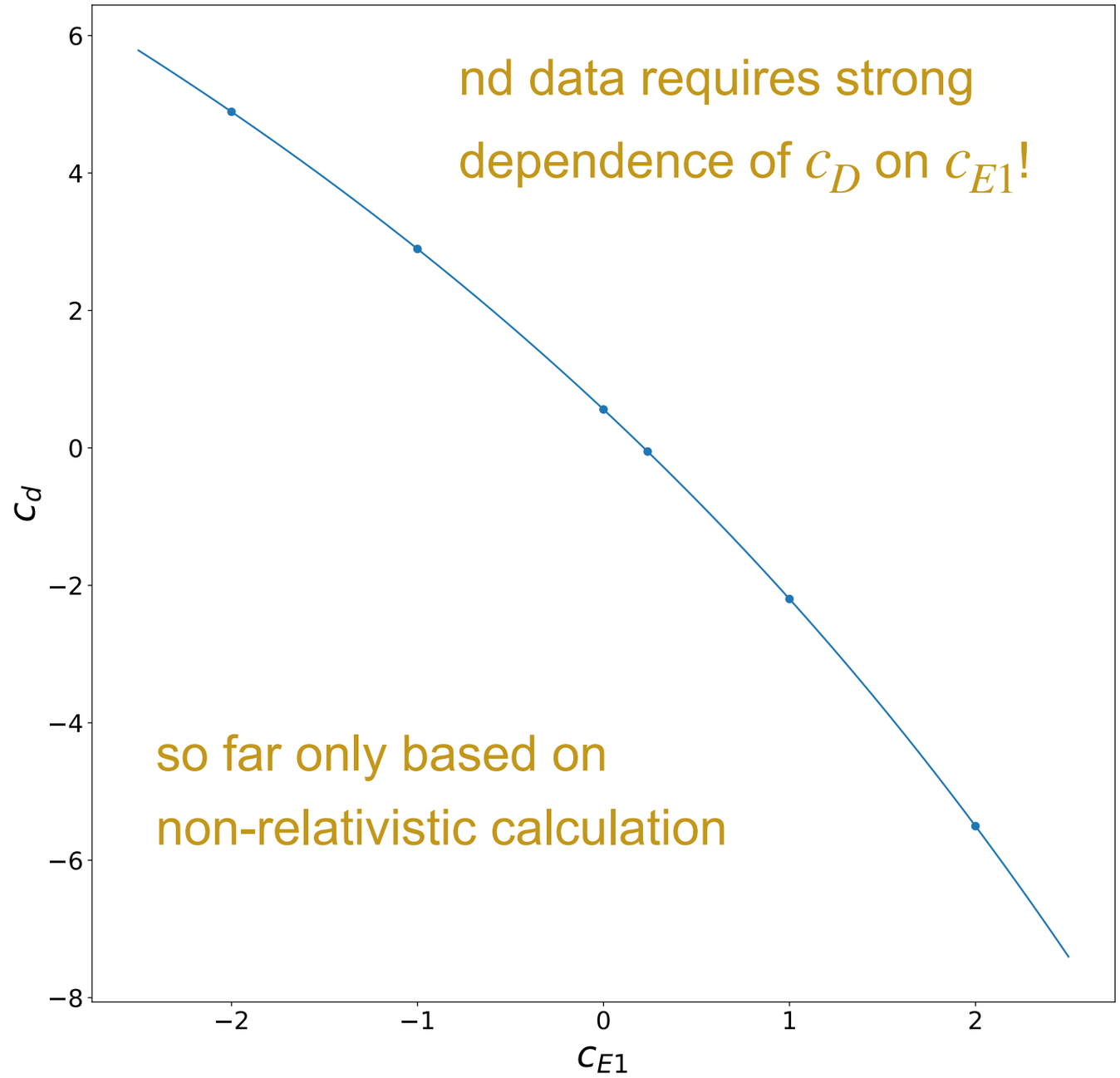
Can adaption of binding energy with c_{E1} help?

Comparison in 4N

promotion of higher order 3NF in order to fix binding energies

c_D, c_E, c_{E1} to be determined from nd cross section minimum, ^{3}He and ^{4}He energies

Starting point:



parameter choice for relativization? fixed c_D or fixed relation? something new?

Comparison in 3N & 4N

potnr	cd	ce	cE1	E[MeV]	H1[MeV]	H2[MeV]	T[MeV]	r(p)[fm]
90.0	-0.053040	-0.177700	0.237540	-28.296151	-28.2925	-28.2830	74.1113	1.43257
92.0	-1.850294	0.161038	0.883122	-28.295970	-28.3117	-28.3164	72.5635	1.42082
92.0	0.559731	-0.213271	1.267017	-28.296000	-28.3298	-28.3350	73.1920	1.42996

potnr	cd	ce	cE1	mt3	E[MeV]	T[MeV]	r(p)[fm]
90.0	-0.053041	-0.177701	0.237540	-1.0	-8.459652	35.7453	1.58015
92.0	-1.850294	0.161038	0.883122	-1.0	-8.456053	35.0089	1.57827
92.0	0.559731	-0.213271	1.267017	-1.0	-8.449512	34.9888	1.58350
90.0	-0.053041	-0.177701	0.237540	1.0	-7.720323	35.0124	1.77165
92.0	-1.850294	0.161038	0.883122	1.0	-7.718653	34.2972	1.76902
92.0	0.559731	-0.213271	1.267017	1.0	-7.718666	34.2947	1.77364

rel. and non-rel. c_{E1} are different

c_D might shift by large values

c_D dependence becomes significant

estimate of rel. corrections is related to c_D determination

Conclusion & Outlook

- relativization might be necessary for high accuracy predictions of matrix elements
  - *boost corrections were treated as 3NF — reasonable?*
    explicit treatment more involved

- good news: relativization works for 3N and 4N bound states

- several code improvements are necessary
  - *iterative solution of LS equation*
  - *use of conserved quantum numbers*
  - *explicit treatment of 3+1 coordinates*

- size of rel. corrections cannot be reliably estimated at this point
- choice of $c_D/c_E/c_{E1}$ is relevant for predictions

- contribution of rel. corrections to radii of heavier nuclei?