Progress in Applications of Basis Light-Front Quantization to QED and QCD

James P. Vary

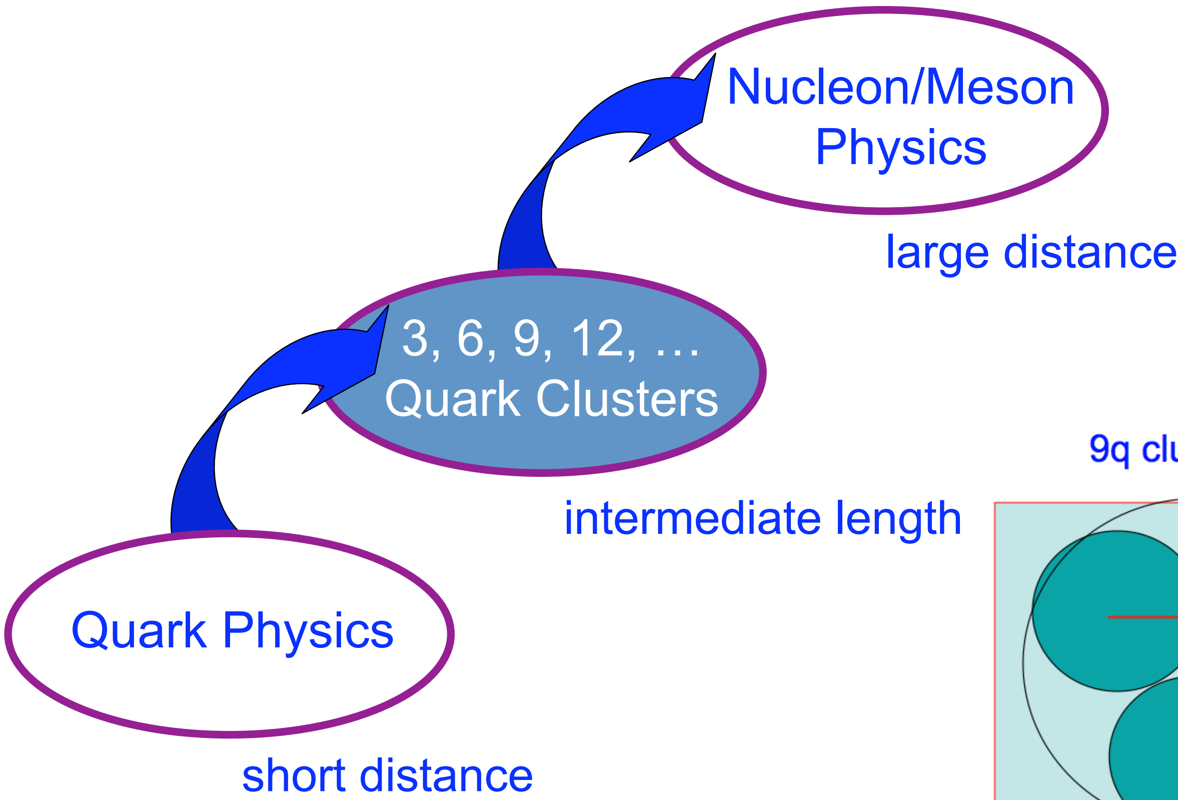
With Xingbo Zhao, Anton Ilderton,
Paul Wiecki, Yang Li,
Heli Honkanen, Pieter Maris,
Stanley J. Brodsky

Department of Physics and Astronomy
Iowa State University
Ames, USA

LC-2013 Meeting, Skiathos, Greece, May 20-24, 2013

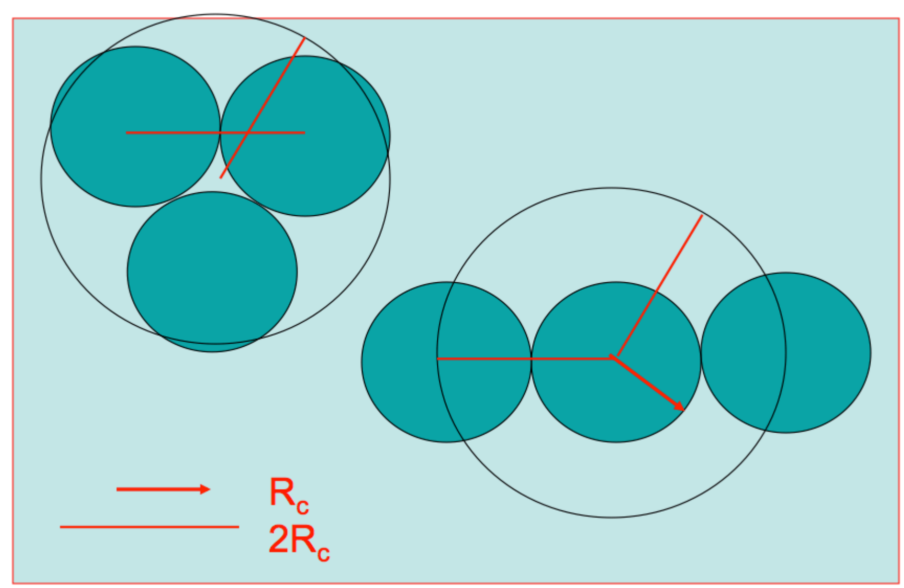
Under what conditions do we require a quark-based description on nuclear structure?

“Quark Percolation in Cold and Hot Nuclei”



Probes with Q > 1 GeV/c
Spin content of the proton
Nuclear form factors
DIS on nuclei – Bjorken x > 1
Nuclear Equation of State

9q cluster at geometrical limits of formation



H.J. Pirner and J.P. Vary,
Phys. Rev. C. **84**, 015201(2011);
arXiv: nucl-th/1008.4962

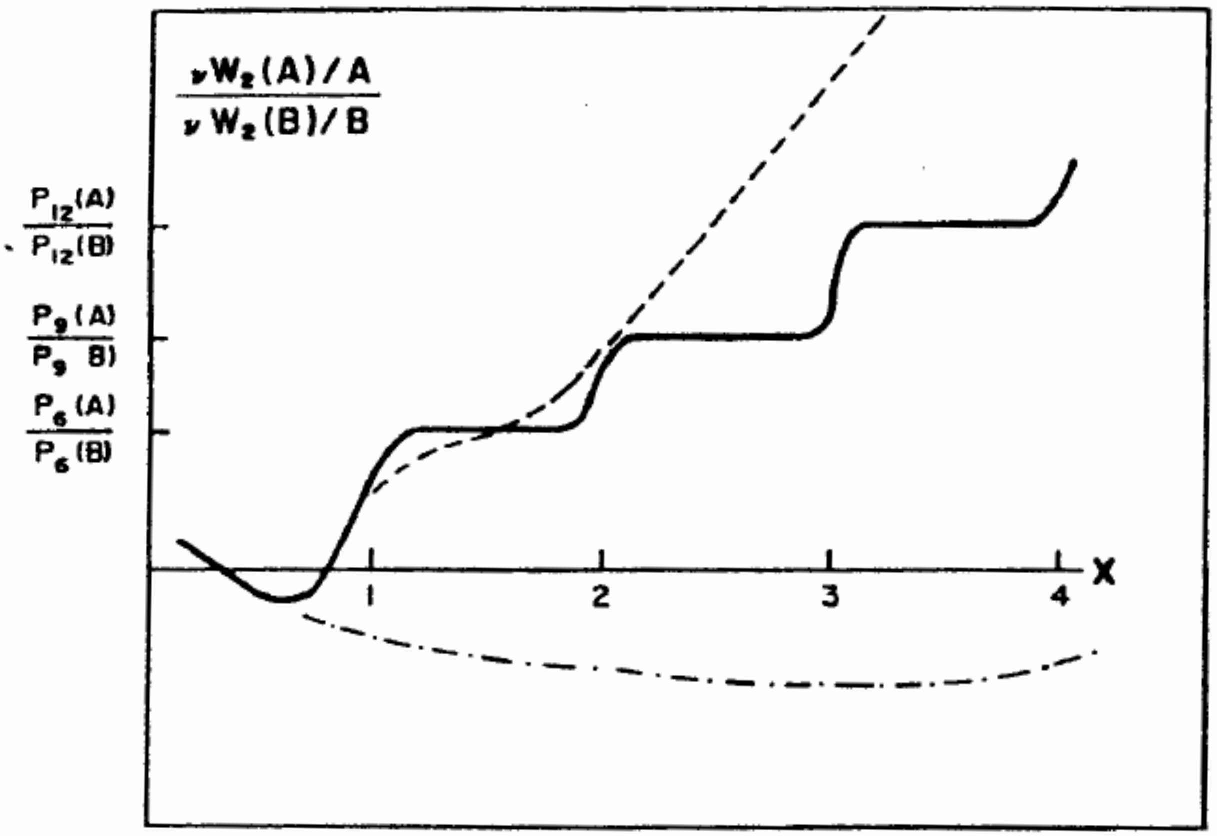
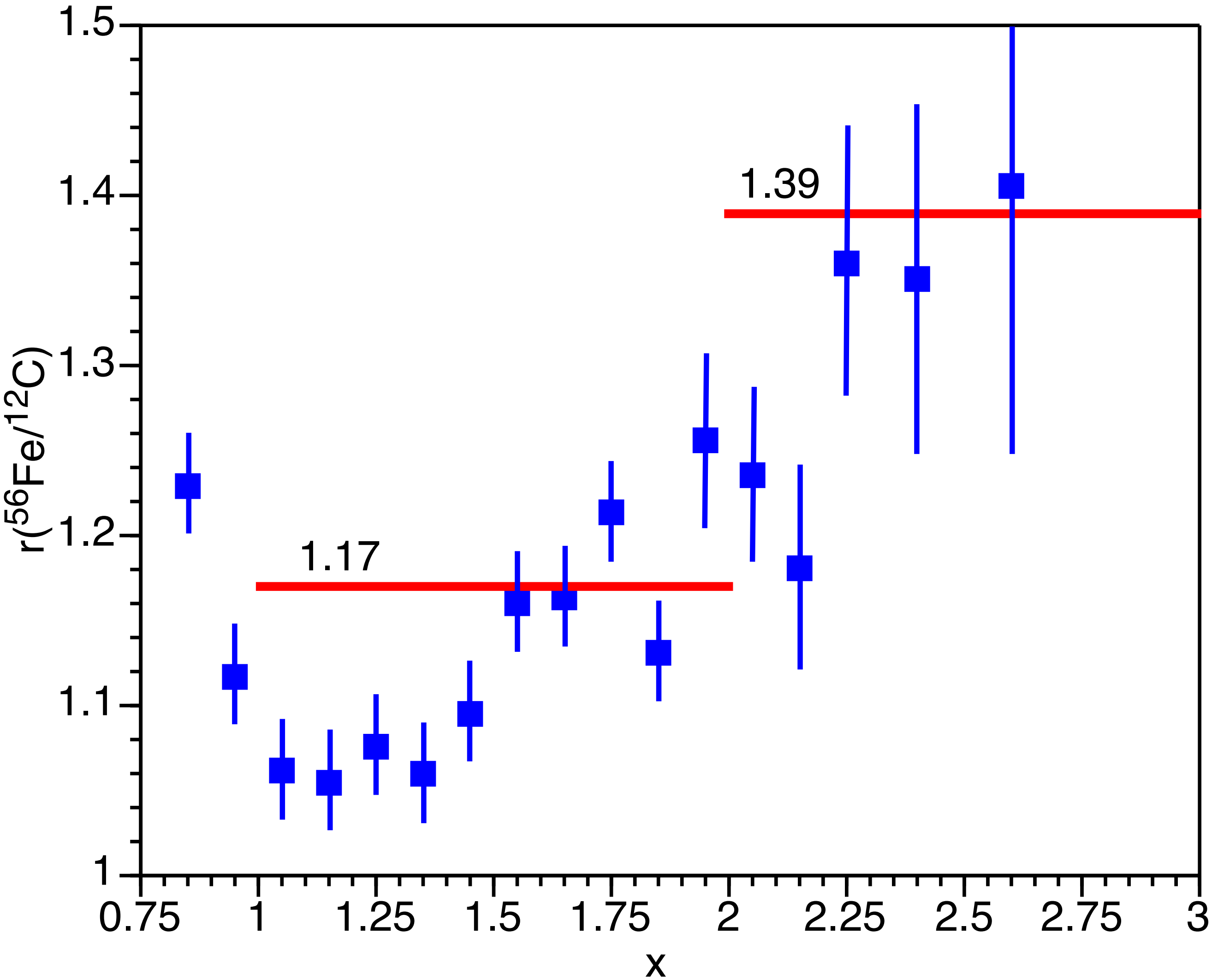


Fig. 2. Characteristic behaviour of the ratio of nuclear structure functions per nucleon for different models over a wide kinematic range of x. The QCM gives the solid curve. The dashed curve is due to the model of reference 22. The dashed-dot curve approximates the predictions of references 23 and 24.

J.P. Vary, Proc. VII Int'l Seminar on High Energy Physics Problems, "Quark Cluster Model of Nuclei and Lepton Scattering Results," Multiquark Interactions and Quantum Chromodynamics, V.V. Burov, Ed., Dubna #D-1, 2-84-599 (1984) 186 [staircase function for x > 1]

See also: Proceedings of HUGS at CEBAF1992, & many conf. proceedings

Comparison between Quark-Cluster Model and JLAB data



Data: K.S. Egiyan, et al., Phys. Rev. Lett. **96**, 082501 (2006)

Theory: H.J. Pirner and J.P. Vary, Phys. Rev. Lett. **46**, 1376 (1981)
and Phys. Rev. C **84**, 015201 (2011); nucl-th/1008.4962;
M. Sato, S.A. Coon, H.J. Pirner and J.P. Vary, Phys. Rev. C **33**, 1062 (1986)

New Measurements of High-Momentum Nucleons and Short-Range Structures in Nuclei

N. Fomin,[1,2,3] J. Arrington,[4] R. Asaturyan,[5,*] F. Benmokhtar,[6] W. Boeglin,[7] P. Bosted,[8] A. Bruell,[8] M. H. S. Bukhari,[9] M. E. Christy,[8] E. Chudakov,[8] B. Clasie,[10] S. H. Connell,[11] M. M. Dalton,[3] A. Daniel,[9] D. B. Day,[3] D. Dutta,[12,13] R. Ent,[8] L. El Fassi,[4] H. Fenker,[8] B. W. Filippone,[14] K. Garrow,[15] D. Gaskell,[8] C. Hill,[3] R. J. Holt,[4] T. Horn,[6,8,16] M. K. Jones,[8] J. Jourdan,[17] N. Kalantarians,[9] C. E. Keppel,[8,18] D. Kiselev,[17] M. Kotulla,[17] R. Lindgren,[3] A. F. Lung,[8] S. Malace,[18] P. Markowitz,[7] P. McKee,[3] D. G. Meekins,[8] H. Mkrtchyan,[5] T. Navasardyan,[5] G. Niculescu,[19] A. K. Opper,[20] C. Perdrisat,[21] D. H. Potterveld,[4] V. Punjabi,[22] X. Qian,[13] P. E. Reimer,[4] J. Roche,[20,8] V. M. Rodriguez,[9] O. Rondon,[3] E. Schulte,[4] J. Seely,[10] E. Segbefia,[18] K. Slifer,[3] G. R. Smith,[8] P. Solvignon,[8] V. Tadevosyan,[5] S. Tajima,[3] L. Tang,[8,18] G. Testa,[17] R. Trojer,[17] V. Tvaskis,[18] W. F. Vulcan,[8] C. Wasko,[3] F. R. Wesselmann,[22] S. A. Wood,[8] J. Wright,[3] and X. Zheng[3,4]

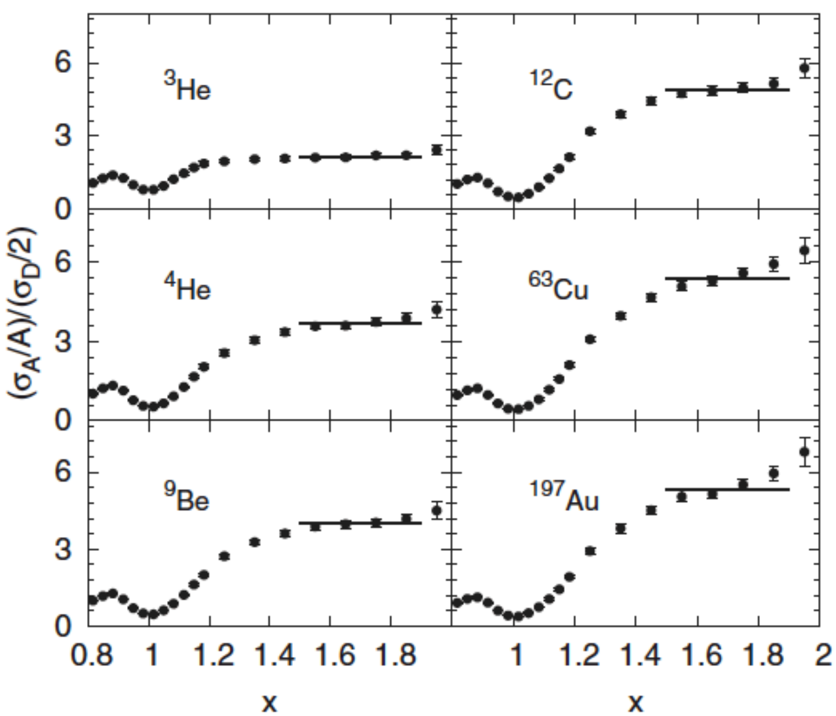


FIG. 2. Pernucleon cross section ratios vs x at $\theta_e = 18°$.

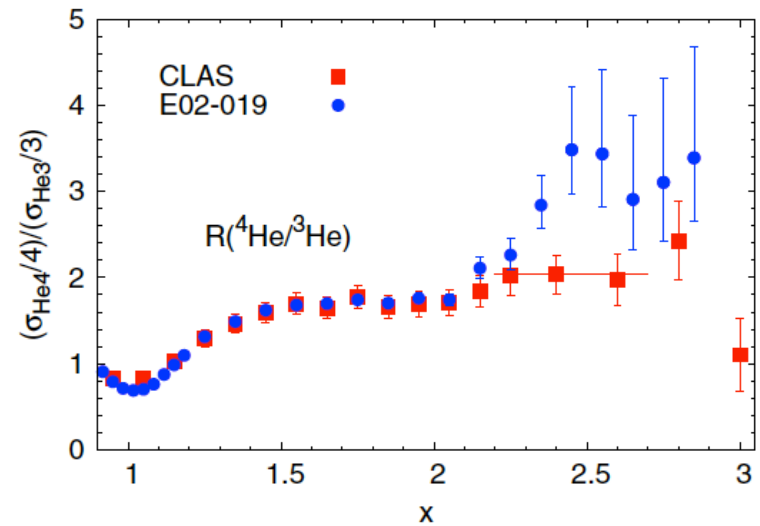


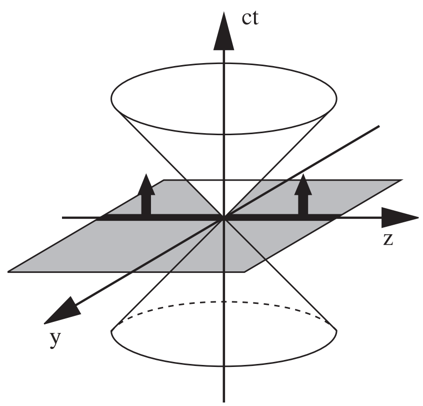
FIG. 3 (color online). The ^{4}He/^{3}He ratios from E02-019 ($Q^2 \approx 2.9$ GeV^2) and CLAS ($\langle Q^2 \rangle \approx 1.6$ GeV^2); errors are combined statistical and systematic uncertainties. For $x > 2.2$, the uncertainties in the ^{3}He cross section are large enough that a one-sigma variation of these results yields an asymmetric error band in the ratio. The error bars shown for this region represent the central 68% confidence level region.

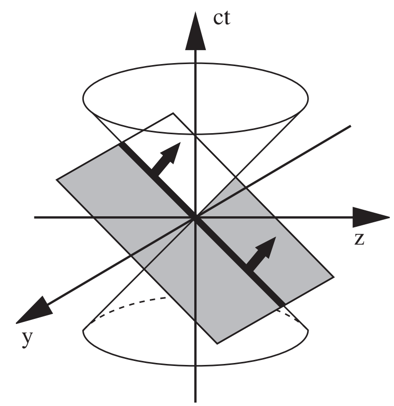
Basis Light-Front Quantization Approach

[Dirac 1949]

- Basic idea: solve generalized wave eq. for quantum field evolution

equal time quantization	VS	light front quantization
$i\frac{\partial}{\partial t}\left\|\varphi(t)\right\rangle = H\left\|\varphi(t)\right\rangle$		$i\frac{\partial}{\partial x^+}\left\|\varphi(x^+)\right\rangle = P_+\left\|\varphi(x^+)\right\rangle$





- Time: $t \equiv x^0$ $\quad$ $t \equiv x^+ = x^0 + x^3$
- Hamiltonian: $H \equiv P^0$ $\quad$ $H \equiv P_+ = \frac{P^0 - P^3}{2}$
- On-shell condition: $P^0 = \sqrt{m^2 + P_\perp^2 + P_3^2}$ $\quad$ $P_+ = \frac{m^2 + P_\perp^2}{2P^+}$

Discretized Light Cone Quantization

Pauli & Brodsky c1985

⇩

Basis Light Front Quantization*

$$\phi(\vec{x}) = \sum_{\alpha} \left[f_{\alpha}(\vec{x}) a_{\alpha}^{+} + f_{\alpha}^{*}(\vec{x}) a_{\alpha} \right]$$

where $\{a_{\alpha}\}$ satisfy usual (anti-) commutation rules.

Furthermore, $f_{\alpha}(\vec{x})$ are arbitrary except for conditions:

Orthonormal: $\int f_{\alpha}(\vec{x}) f_{\alpha'}^{*}(\vec{x}) d^3x = \delta_{\alpha\alpha'}$

Complete: $\sum_{\alpha} f_{\alpha}(\vec{x}) f_{\alpha}^{*}(\vec{x}') = \delta^3(\vec{x} - \vec{x}')$

=> Wide range of choices for $f_{a}(\vec{x})$ and our initial choice is

$$f_{\alpha}(\vec{x}) = Ne^{ik^{+}x^{-}} \Psi_{n,m}(\rho,\varphi) = Ne^{ik^{+}x^{-}} f_{n,m}(\rho) \chi_{m}(\varphi)$$

*J.P. Vary, H. Honkanen, J. Li, P. Maris, S.J. Brodsky, A. Harindranath, G.F. de Teramond, P. Sternberg, E.G. Ng and C. Yang, PRC 81, 035205 (2010). ArXiv:0905:1411

Basis Functions

[Vary et al ’10, Honkanen et al ‘11]

- Optimal basis is the key to numerical efficiency

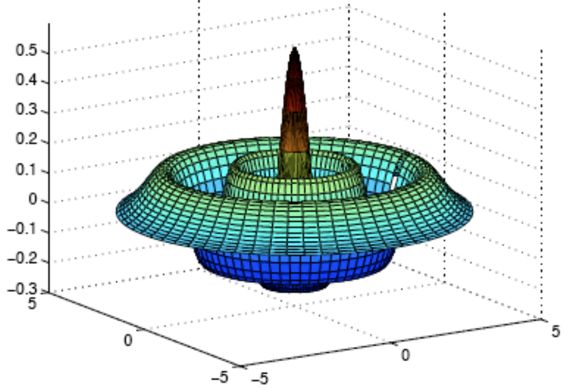
- Optimal basis reflects the symmetries of the underlying theory

- Multi-particle basis is set up by Fock sector expansion
  - e.g., $|e_{\text{physical}}\rangle = a|e\rangle + b|e\gamma\rangle + c|e\gamma\gamma\rangle + d|e\gamma e\bar{e}\rangle + \dots$

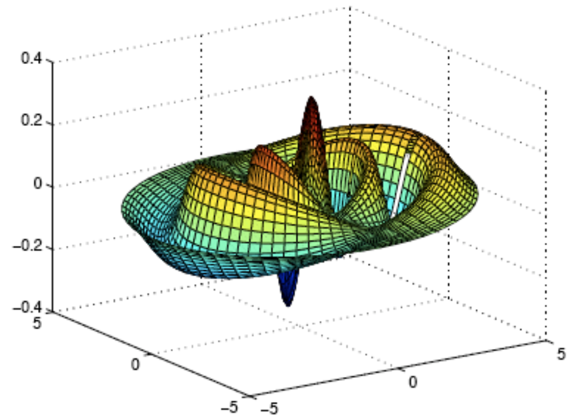
- For individual particles in each Fock sector
  - plane wave basis for longitudinal ($x^- \equiv x^0 - x^3$) direction
  - 2D harmonic oscillator basis for transverse (x^1, x^2) directions
  - reflect the rotational symmetry of QED, QCD… in the transverse plane

Set of transverse 2D HO modes for n=4

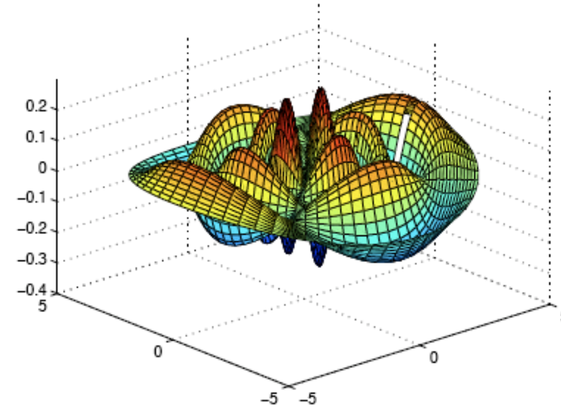
m=0



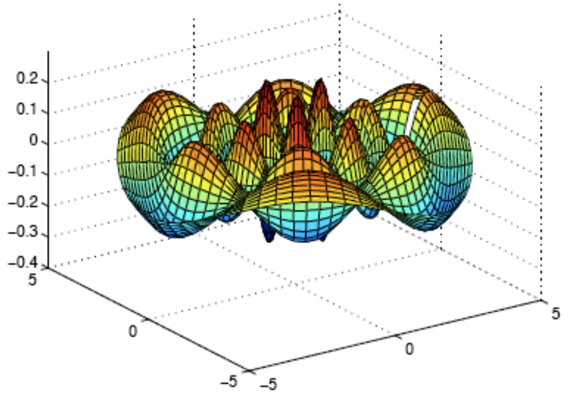
m=1



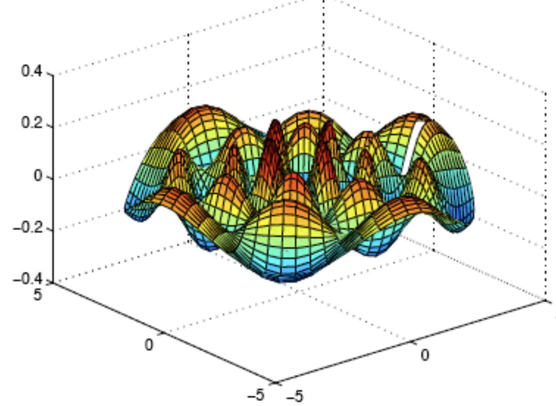
m=2



m=3



m=4



J.P. Vary, H. Honkanen, J. Li, P. Maris, S.J. Brodsky, A. Harindranath,
G.F. de Teramond, P. Sternberg, E.G. Ng and C. Yang, PRC 81, 035205 (2010).
ArXiv:0905:1411

Steps to implement BLFQ

- Enumerate Fock-space basis subject to symmetry constraints
- Evaluate/renormalize/store H in that basis
- Diagonalize (Lanczos)
- Iterate previous two steps for sector-dep. renormalization
- Evaluate observables using eigenvectors (LF amplitudes)
- Repeat previous 4 steps for new regulator(s)
- Extrapolate to infinite matrix limit & remove all regulators
- Compare with experiment or predict new experimental results

Above achieved for QED test case – electron in a trap
H. Honkanen, P. Maris, J.P. Vary, S.J. Brodsky,
Phys. Rev. Lett. 106, 061603 (2011)

Improvements: trap independence, (m,e) renormalization, . . .
X. Zhao, H. Honkanen, P. Maris, J.P. Vary, S.J. Brodsky, in prep'n

Symmetries & Constraints

$$\sum_i b_i = B$$

$$\sum_i e_i = Q$$

$$\sum_i (m_i + s_i) = J_z$$

$$\sum_i k_i = K$$

$$\sum_i \left[2n_i + |m_i| + 1\right] \leq N_{\max}$$

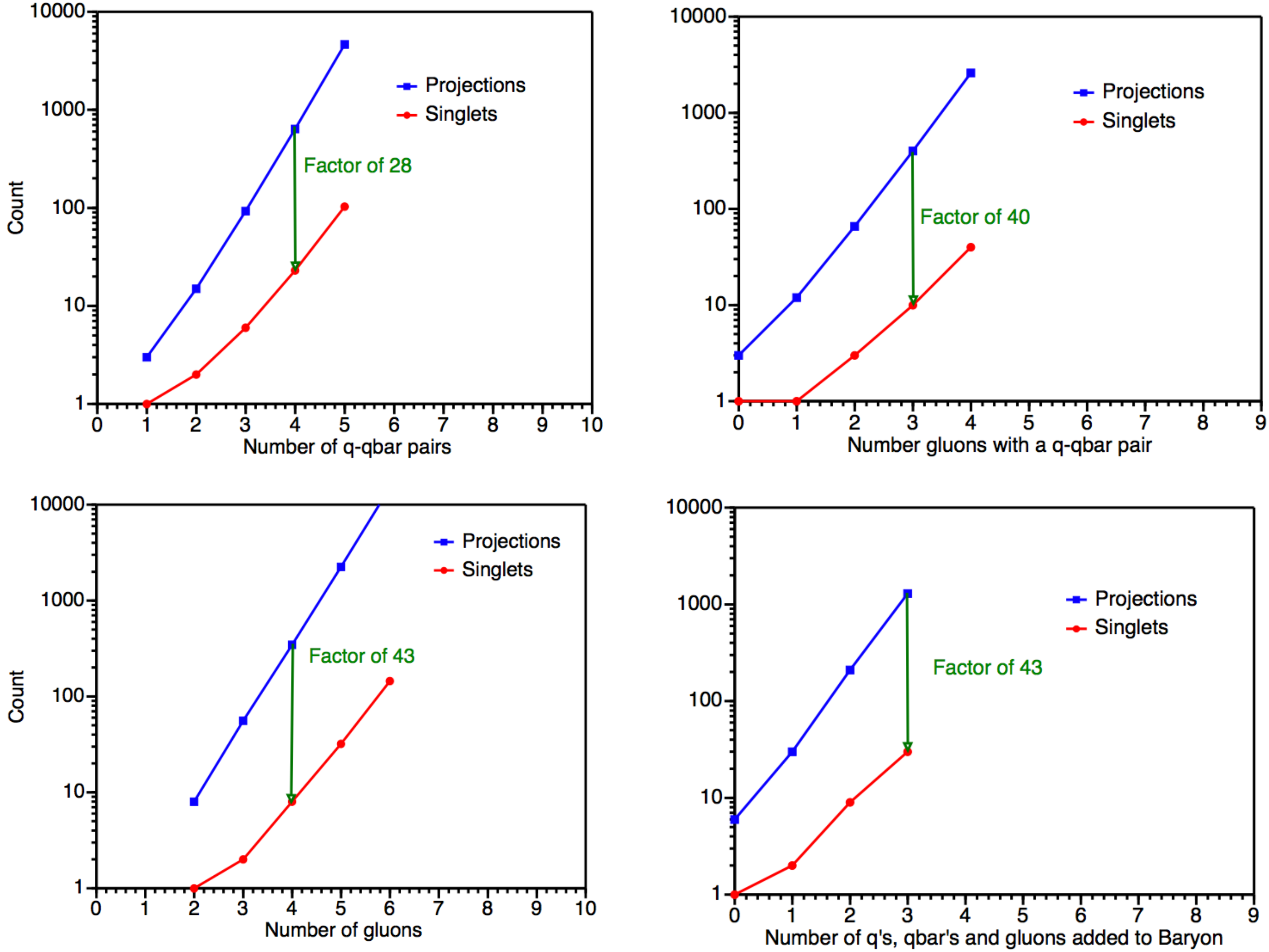
Global Color Singlets (QCD)

Light Front Gauge

Optional - Fock space cutoffs

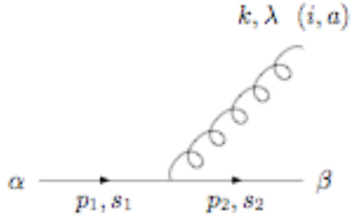
$$\mathrm{H} \rightarrow \mathrm{H} + \lambda \mathrm{H}_{CM}$$

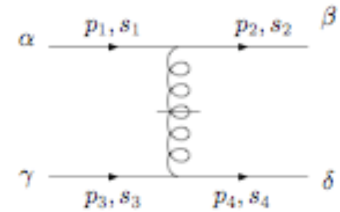
Finite basis regulators



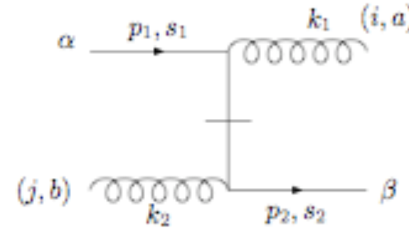
J.P. Vary, H. Honkanen, J. Li, P. Maris, S.J. Brodsky, A. Harindranath,
G.F. de Teramond, P. Sternberg, E.G. Ng and C. Yang, PRC 81, 035205 (2010). ArXiv:0905:1411

Light Front (LF) Hamiltonian defined by its elementary vertices in LF gauge

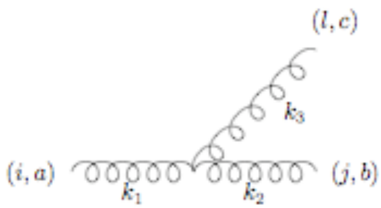


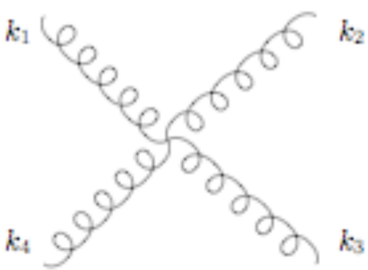


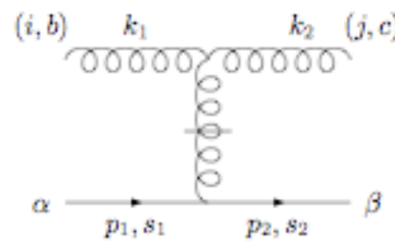
H_{qqgg1}



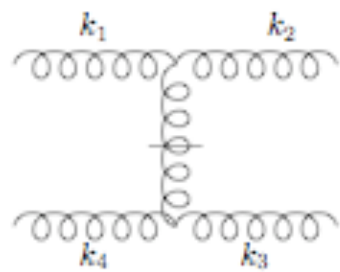
QED & QCD







H_{qqgg2}



QCD

$$
\begin{aligned}
H = &\frac{1}{2}\int d^3x \overline{\widetilde{\psi}}\gamma^+ \frac{(\mathrm{i}\partial^\perp)^2 + m^2}{\mathrm{i}\partial^+}\widetilde{\psi} - A_a^i(\mathrm{i}\partial^\perp)^2 A_{ia} \\
&- \frac{1}{2}g^2\int d^3x \mathbf{Tr}\left[\widetilde{A}^\mu, \widetilde{A}^\nu\right]\left[\widetilde{A}_\mu, \widetilde{A}_\nu\right] \\
&+ \frac{1}{2}g^2\int d^3x \overline{\widetilde{\psi}}\gamma^+ T^a\widetilde{\psi}\frac{1}{(\mathrm{i}\partial^+)^2}\overline{\widetilde{\psi}}\gamma^+ T^a\widetilde{\psi} \\
&- g^2\int d^3x \overline{\widetilde{\psi}}\gamma^+\left(\frac{1}{(\mathrm{i}\partial^+)^2}\left[\mathrm{i}\partial^+\widetilde{A}^\kappa, \widetilde{A}_\kappa\right]\right)\widetilde{\psi} \\
&+ g^2\int d^3x \mathbf{Tr}\left(\left[\mathrm{i}\partial^+\widetilde{A}^\kappa, \widetilde{A}_\kappa\right]\frac{1}{(\mathrm{i}\partial^+)^2}\left[\mathrm{i}\partial^+\widetilde{A}^\kappa, \widetilde{A}_\kappa\right]\right) \\
&+ \frac{1}{2}g^2\int d^3x \overline{\widetilde{\psi}}\widetilde{\not{A}}\frac{\gamma^+}{\mathrm{i}\partial^+}\widetilde{\not{A}}\widetilde{\psi} \\
&+ g\int d^3x \overline{\widetilde{\psi}}\widetilde{\not{A}}\widetilde{\psi} \\
&+ 2g\int d^3x \mathbf{Tr}\left(\mathrm{i}\partial^\mu\widetilde{A}^\nu\left[\widetilde{A}_\mu, \widetilde{A}_\nu\right]\right)
\end{aligned}
$$

Regularization and Renormalization Schemes

1. Basis space regulators (2-D HO params, K)
2. Additional Fock space truncations (if any)
3. Counterterms identified/tested*
4. Sector-dependent renormalization**
5. SRG, OLS, . . . Adapted to BLFQ

*D. Chakrabarti, A. Harindranath and J.P. Vary,
"A Study of q-qbar States in Transverse Lattice QCD
Using Alternative Fermion Formulations,"
Phys. Rev. D **69**, 034502 (2004); hep-ph/0309317

**V. A. Karmanov, J.-F. Mathiot, and A. V. Smirnov,
Phys. Rev. D **77**, 085028 (2008);
and new paper - arXiv:1204.3257

Evaluate Electron g-2 with BLFQ Approach

[Honkanen, et al, 2011]

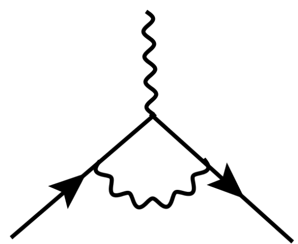
- Electron anomalous magnetic moment

$$a_e \equiv \frac{g-2}{2}$$

- Leading contribution to a_e is from QED

[Schwinger 1948]

$$a_e = \frac{\alpha}{2\pi} \quad \left(\alpha = \frac{1}{137}\right)$$

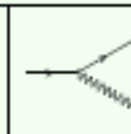
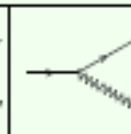
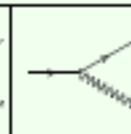
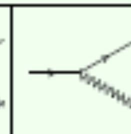
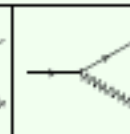
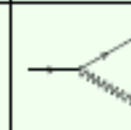
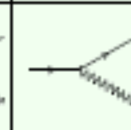
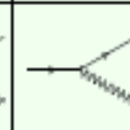
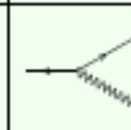
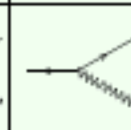
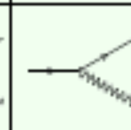
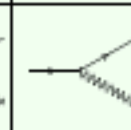
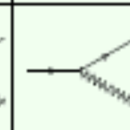
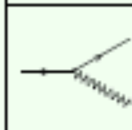
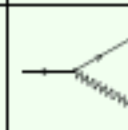
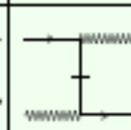
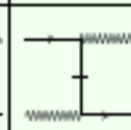
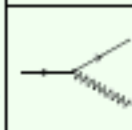
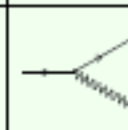
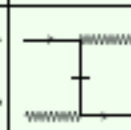
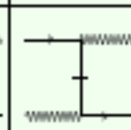
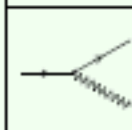
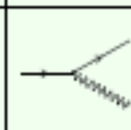
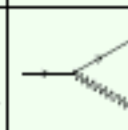
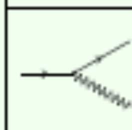
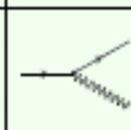
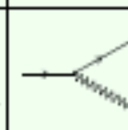
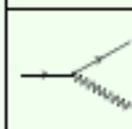
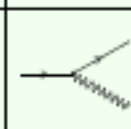
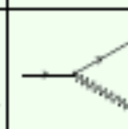


- a_e is electron Pauli form factor at zero-moment transfer limit:

$$a_e = F_2(q^2 \to 0)$$

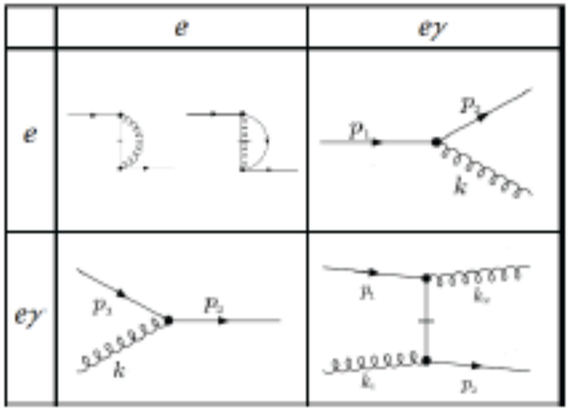
- In BLFQ, $a_e = \left\langle e_{physical} \middle| \hat{F}_2(q^2 \to 0) \middle| e_{physical} \right\rangle$

Matrix Example: $N_{max} = 3, K = 2$

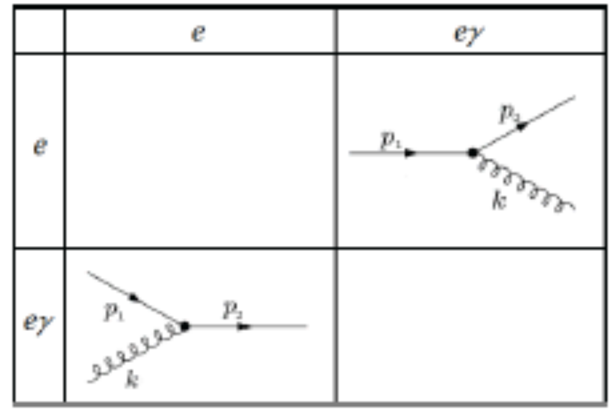
e: ↑, n=m=0	0	0	0					
e: ↓, n=0, m=1	0	0	0	0	0			
e: ↑, n=1, m=0	0	0	0					
e: ↑, n=0, m=-1 γ: ↑, n=m=0		0				0	0	0
e: ↑, n=m=0 γ: ↑, n=0, m=-1		0				0	0	0
e: ↓, n=m=0 γ: ↑, n=m=0				0	0	0	0	0
e: ↑, n=m=0 γ: ↓, n=0, m=1				0	0	0	0	0
e: ↑, n=0, m=1 γ: ↓, n=m=0				0	0	0	0	0

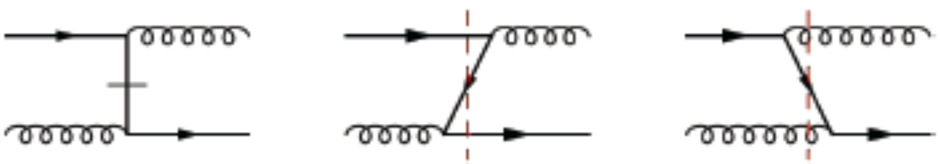
$$|e_{\text{ph}}\rangle = |e\rangle + |e\gamma\rangle$$

$P_{+}^{int} =$



$\Longrightarrow$ $P_{+}^{int} =$

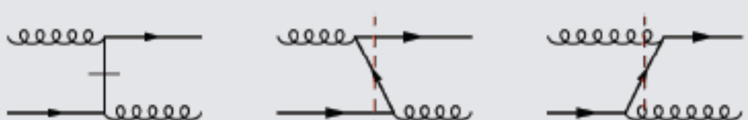




To cancel out the small-x divergence in the instantaneous vertex needs diagrams from $e\gamma\gamma + ee\bar{e}$ sectors.

The instantaneous self-energy vertices can be absorbed into the mass term

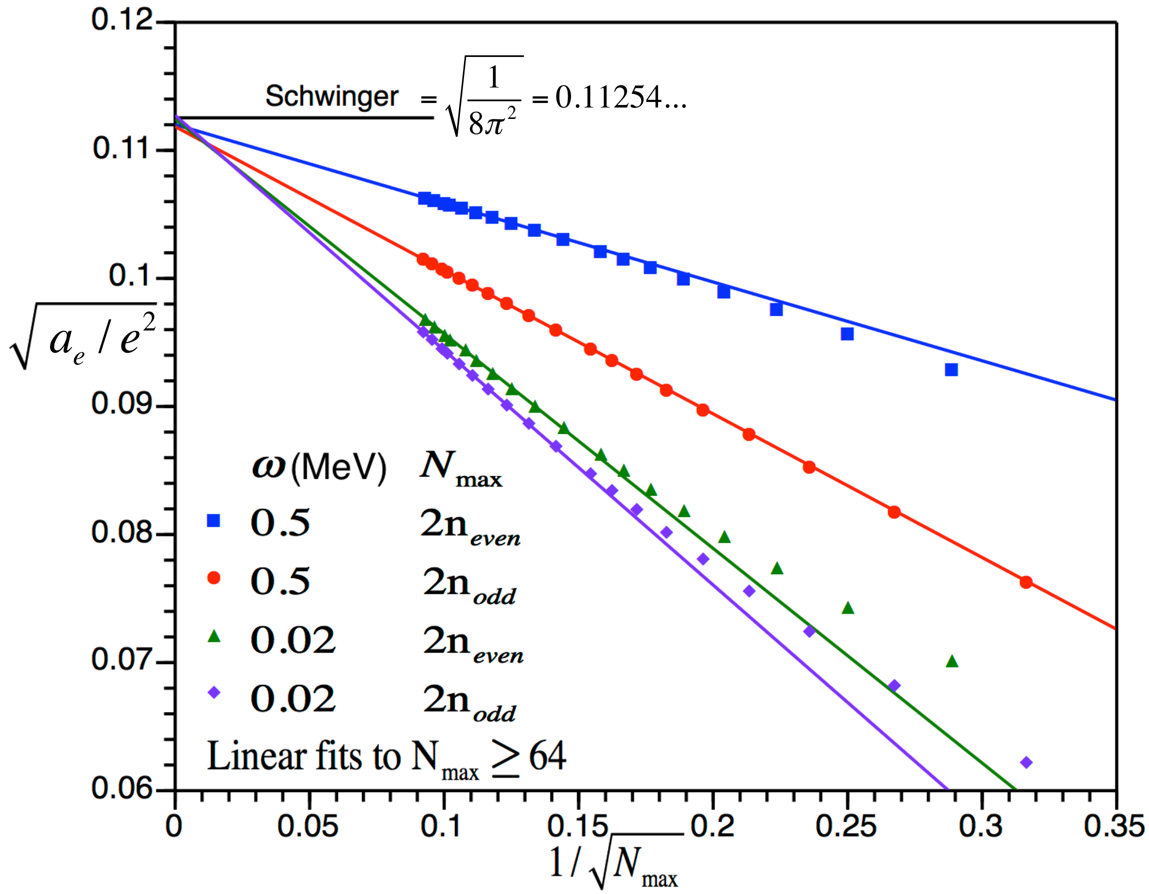
- **Neglect instantaneous interactions when corresponding dynamical exchange is not present in model space**



This may be generalized to higher Fock space basis

Numerical Results for Electron g-2

[X. Zhao, H. Honkanen, P. Maris, J.P. Vary, S.J. Brodsky, in preparation as major update to:
H. Honkanen, P. Maris, J.P. Vary, S.J. Brodsky, Phys. Rev. Lett. 106, 061603 (2011)]



- As Nmax → ∞, results approach Schwinger result
- Less than 1% deviation from Schwinger's result (by linear extrapl.)
- Convergence over wide range of ω's (by a factor of 25!)

- AdS/QCD basis
  - Let $\boldsymbol{q}^\perp = \frac{\boldsymbol{p}^\perp}{\sqrt{x}}$, its conjugate coordinate is $\boldsymbol{s}^\perp = \sqrt{x}\boldsymbol{r}$;

$$\boldsymbol{s}^\perp_{\text{rel}} = \sqrt{x_1 x_2}(\boldsymbol{r}^\perp_1 - \boldsymbol{r}^\perp_2)$$

  - AdS/QCD and LF holography
    As shown by Brodsky and de Teramond, one can use LF holography to translate the AdS/QCD eigenmodes to LF wavefunctions.

BLFQ	LF Holography	"soft-wall" AdS/QCD
relative coordinate $\boldsymbol{s}^\perp_{\text{rel}}$	*impact parameter* ζ	*the 5th dimension* z
confining potential $V_{\text{con}}(s) \propto s^2_{\text{rel}}$	*effective potential* $U(\zeta) \propto \zeta^2 + \text{const.}$	*background dilaton field* $\Phi(z) \propto z^2$
basis function $\phi^m_n(\boldsymbol{s}^\perp_{\text{rel}})$	*eigenstate LFWF* $\phi^m_n(\zeta)$	*string mode amplitude* $\phi^m_n(z)$

$$\Phi_n^m(x, \boldsymbol{p}^\perp) = \mathcal{N} e^{im\theta} \rho^{|m|} e^{-\frac{\rho^2}{2x}} L_n^{|m|}\left(\frac{\rho^2}{x}\right), \quad \rho = |\boldsymbol{p}^\perp|/b, \theta = \arg \boldsymbol{p}^\perp$$
$$\equiv \phi_n^m\left(\frac{\boldsymbol{p}^\perp}{\sqrt{x}}\right)$$

- use single particle basis to build many-body basis;
- generalized 2D Talmi-Moshinsky (TM) transform

$$|n_1 m_1 x_1; n_2 m_2 x_2\rangle = \sum_{NMnm} (NMnm|n_1 m_1 n_2 m_2)|_\delta \, |NMX; nmx\rangle$$

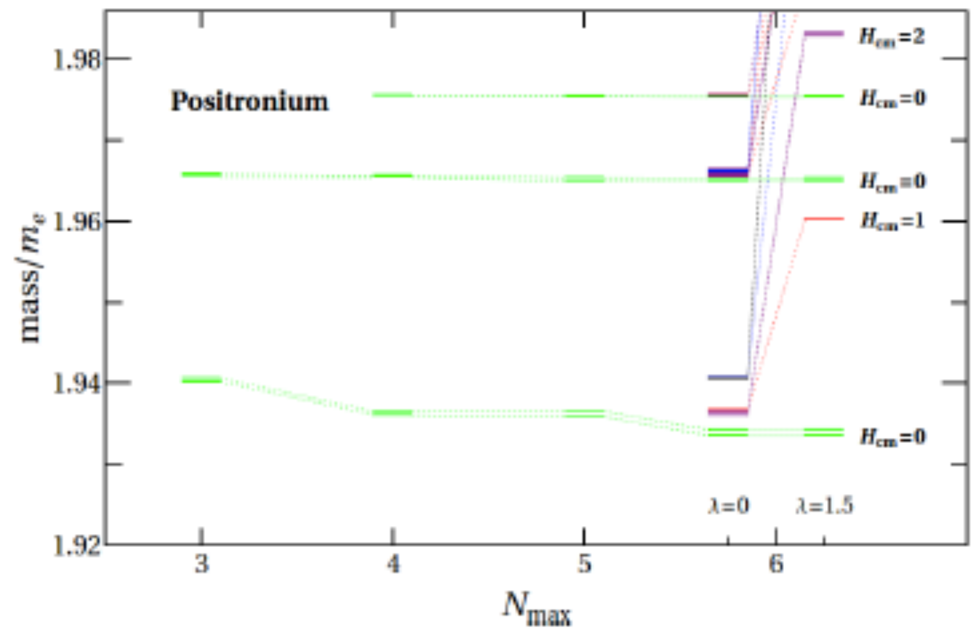
$$X = x_1 + x_2, x = \frac{x_1 x_2}{x_1 + x_2}, \delta = \arctan\sqrt{\frac{x_2}{x_1}}$$

  - closed-form expression for TM coefficients via generating function technique;
  - compatible with $N_{\max}$ truncation: $2n_1 + |m_1| + 2n_2 + |m_2| = 2N + |M| + 2n + |m|$
  - only finite terms in the sum;

- Exact factorization of CM motion in $N_{\max}$ truncation:

$$|\Psi\rangle = |\varphi_{intr}\rangle \otimes |\Phi_{\text{CM}}\rangle$$

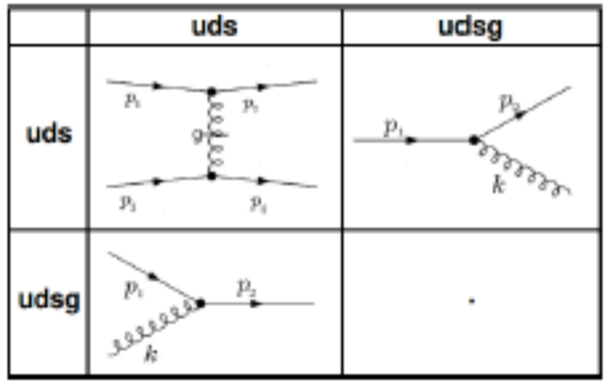
  - not necessarily true for arbitrary finite truncated basis space;
  - not true for $\phi_n^m(\boldsymbol{p}^\perp)$ with $N_{\max}$ truncation;
  - apply Lagrange multiplier method to lift the non-zero $\boldsymbol{P}^\perp$ motion; $H \to H + \lambda H_{\text{cm}}$ with $H_{\text{cm}} = \boldsymbol{P}^2 + b^4\boldsymbol{R}^2 - 2b^2$;



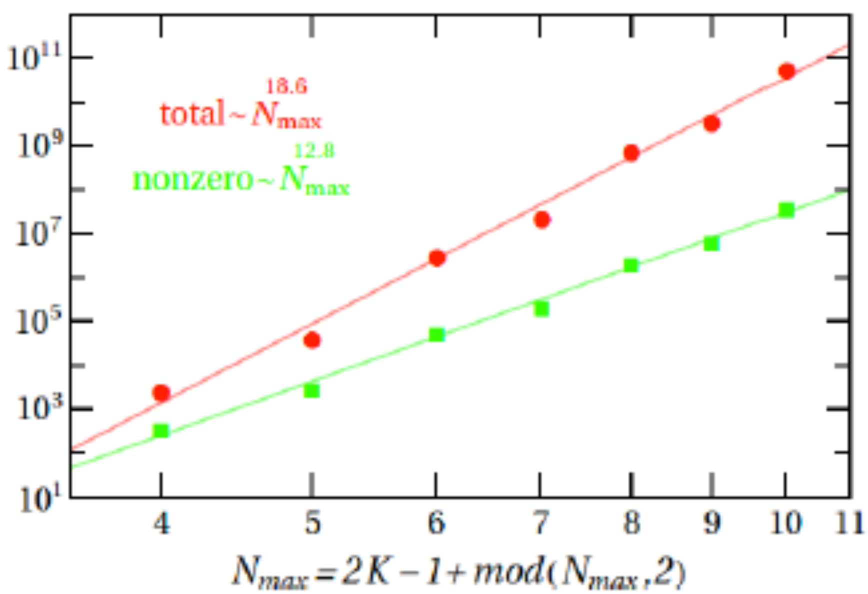
Computational Aspect: a case study

Λ baryon problem:

$$P_+^{int} =$$

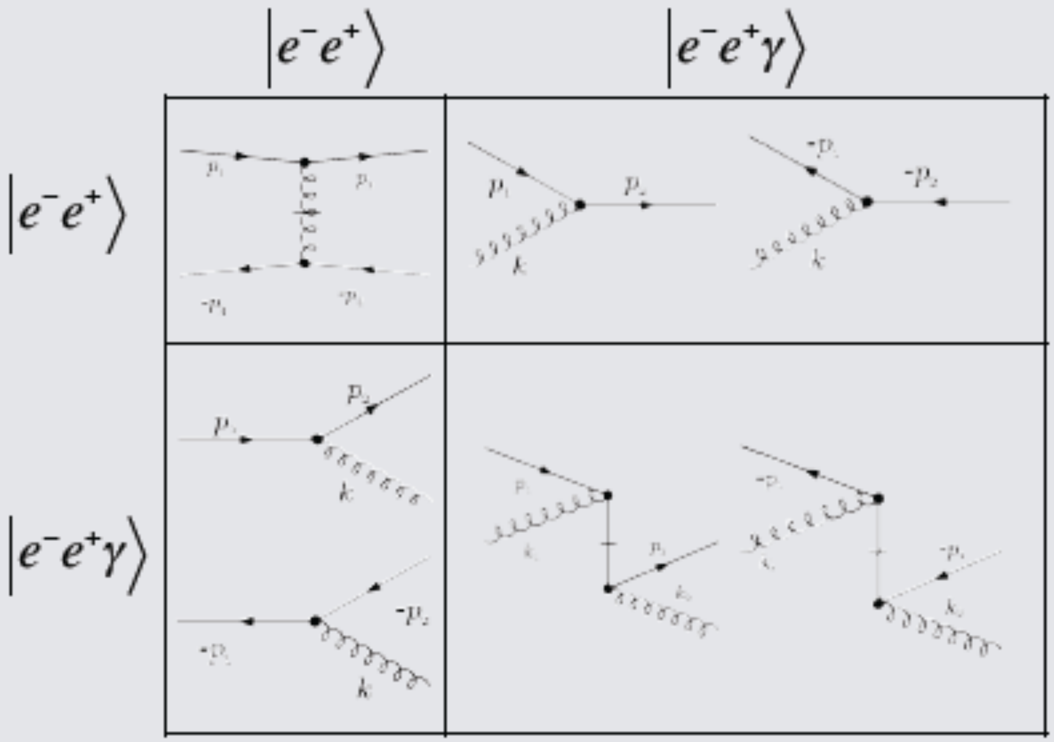


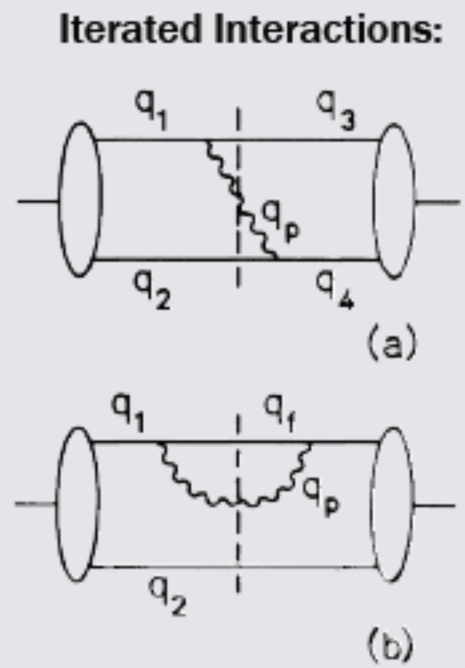
Sparsity of the matrix:



Colorlines: red, total number of matrix elements; green, number of nonzero matrix elements.
K is a half integer and $2K$ is odd;

Positronium





- **Neglect instantaneous interactions when corresponding dynamical exchange is not present in model space**

- **Projection operators P and Q**

$$P^2 \left|\Psi_i\right\rangle = M_i^2 \left|\Psi_i\right\rangle$$

$$H \equiv P^2$$

$$PHP \equiv H_{00}$$
$$PHQ \equiv H_{01}$$
$$QHP \equiv H_{10}$$
$$QHQ \equiv H_{11}$$

	$\left|e^-e^+\right\rangle$	$\left|e^-e^+\gamma\right\rangle$
$\left|e^-e^+\right\rangle$	*PHP*	*PHQ*
$\left|e^-e^+\gamma\right\rangle$	*QHP*	*QHQ*

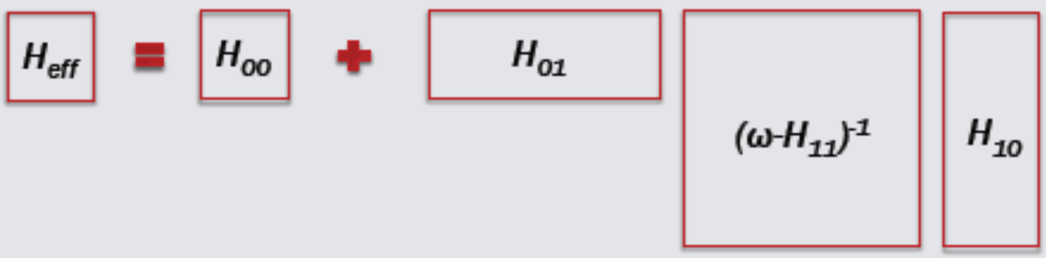
- **Eigenvalue equation can be rewritten:**

$$H_{\text{eff}}(\omega)\left|\psi_i(\omega)\right\rangle_0 = \tilde{M}_i^2(\omega)\left|\psi_i(\omega)\right\rangle_0$$

$$H_{\text{eff}}(\omega) \equiv H_{00} + H_{01}\frac{1}{\omega - H_{11}}H_{10}$$

$$\omega = M_i^2$$

$$H_{\text{eff}}(\omega) \equiv H_{00} + H_{01}\frac{1}{\omega - H_{11}}H_{10}$$



- **In special case where H_{11} is diagonal**

$$\left\langle f\right|H_{\text{eff}}(\omega)\left|i\right\rangle = \left\langle f\right|H_{00}\left|i\right\rangle + \sum_{|n\rangle}\frac{\left\langle f\right|H_{01}\left|n\right\rangle\left\langle n\right|H_{10}\left|i\right\rangle}{\omega - \left\langle n\right|H_{11}\left|n\right\rangle}$$

- **Resolvent operator $(\omega-H_{11})^{-1}$ not diagonal in H.O. basis**
  - **Work in momentum space initially**

- **Consider just second term of effective interaction**

$$\langle f|H^{(2)}_{eff}(\omega)|i\rangle = \sum_{|n\rangle} \frac{\langle f|H_{01}|n\rangle\langle n|H_{10}|i\rangle}{\omega - E_n}$$

- **Sum reduces to sum over polarization states of photon**

$$\sum_{\lambda_g} \varepsilon_\mu(k_g, \lambda_g)\, \varepsilon^*_\nu(k_g, \lambda_g) = -g_{\mu\nu} + (k_{g,\mu}\eta_\nu + k_{g,\nu}\eta_\mu)/k^\kappa_g \eta_\kappa$$

$$\eta^\mu = (\eta^+, \eta_\perp, \eta^-) = (0, 0_\perp, 2)$$

$$;\ \eta^2 \equiv \eta^\mu \eta_\mu = 0$$

$$k\eta = k^+$$

- **Expected cancellation is achieved only if:**

$$\omega = \frac{E_i + E_f}{2}$$

With this choice the divergence of the instantaneous graph is cancelled exactly by divergent part of the effective interaction. Henceforward, these cancelling divergences are dropped from the calculations.

Small x divergences

- **Final result**

$$\langle f | H_{eff}^{(2)}\left(\omega = \tfrac{E_i+E_f}{2}\right) | i \rangle = \alpha \frac{\delta^{x_1'+x_2'}_{x_1+x_2}}{K} \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{d^2 q_1'}{(2\pi)^2} \frac{d^2 q_2'}{(2\pi)^2} (2\pi)^2 \delta^{(2)}\left(\sqrt{x_1} q_1 + \sqrt{x_2} q_2 - \sqrt{x_1'} q_1' + \sqrt{x_2'} q_2'\right)$$

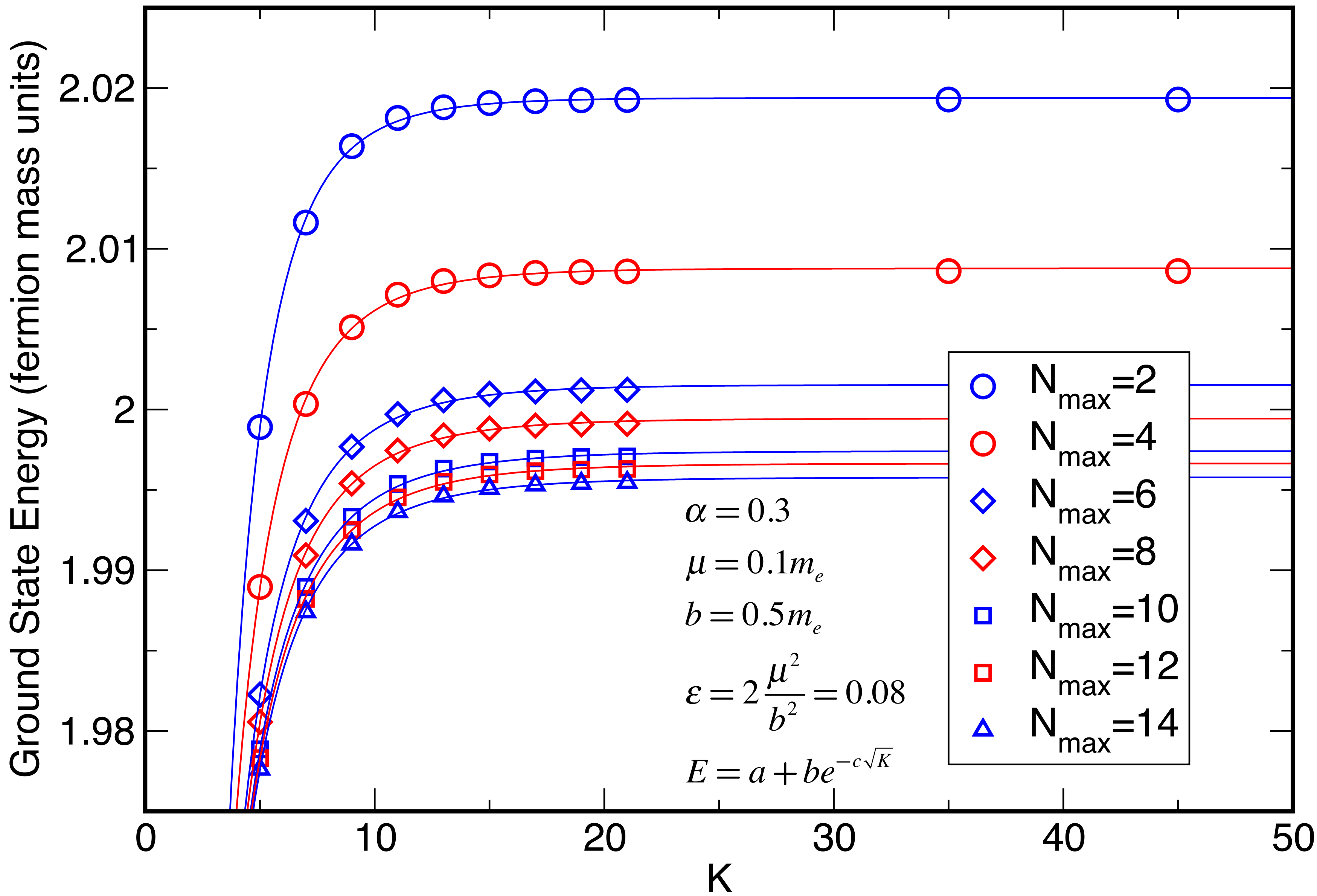
$$\cdot \frac{\Psi^{m_1}_{n_1}(q_1)\Psi^{m_2}_{n_2}(q_2)\Psi^{*m_1'}_{n_1'}(q_1')\Psi^{*m_2'}_{n_2'}(q_2')\bar{u}(1')\gamma^\mu u(1)\bar{v}(2)\gamma_\mu v(2')}{\frac{(x_1-x_1')}{2}\left[\left(\frac{x_1 q_1^2+m^2}{x_1} + \frac{x_1' q_1'^2+m^2}{x_1'} - \frac{\left(\sqrt{x_1}q_1-\sqrt{x_1'}q_1'\right)^2+\mu^2}{x_1-x_1'}\right) - \left(\frac{x_2 q_2^2+m^2}{x_2} + \frac{x_2' q_2'^2+m^2}{x_2'} - \frac{\left(\sqrt{x_2}q_2-\sqrt{x_2'}q_2'\right)^2+\mu^2}{x_2-x_2'}\right)\right]}$$

$$I = -\frac{1}{x_1+x_2}\cdot\frac{1}{2\pi}\cdot\delta^{m_1'+m_2'}_{m_1+m_2}\cdot\sum (TM)(TM)(TM)\int dx dy \frac{e^{-(x+y)/2}L(x)L(y)}{(\alpha-\beta)^2 x + (\alpha+\beta)^2 y + \frac{\Delta}{b^2} + \varepsilon}$$

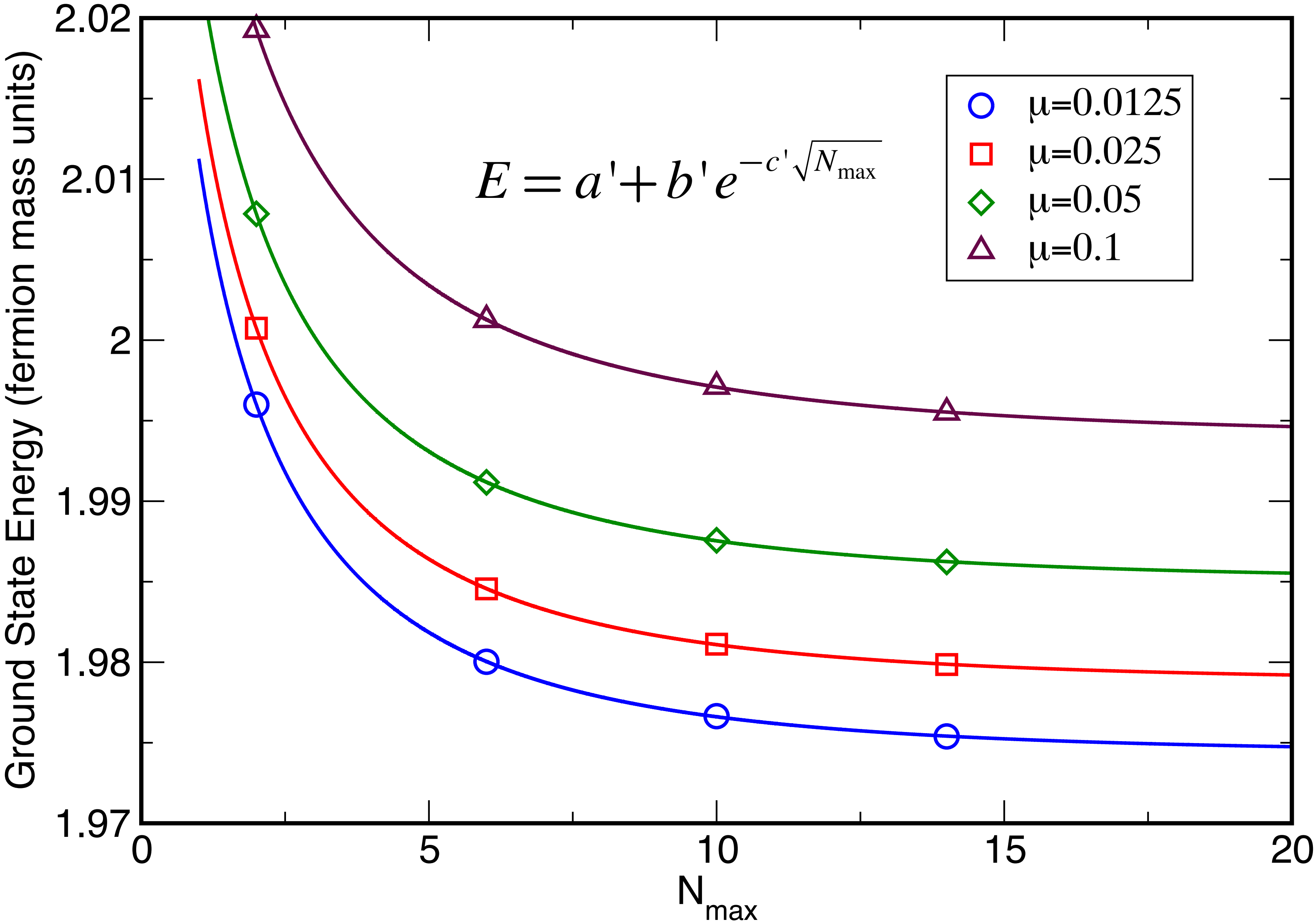
$$\alpha \equiv \sqrt{\tfrac{x_1' x_2}{x_1+x_2}} \qquad \Delta \equiv m^2 (x_1 - x_1')^2 \left[\tfrac{1}{x_1' x_1} + \tfrac{1}{x_2' x_2}\right] \geq 0 \qquad \varepsilon \equiv 2\frac{\mu^2}{b^2}$$

$$\beta \equiv \sqrt{\tfrac{x_2' x_1}{x_1+x_2}}$$

- **3-fold summation**
- **2D Numerical Integral**
- **Infrared Cutoff**

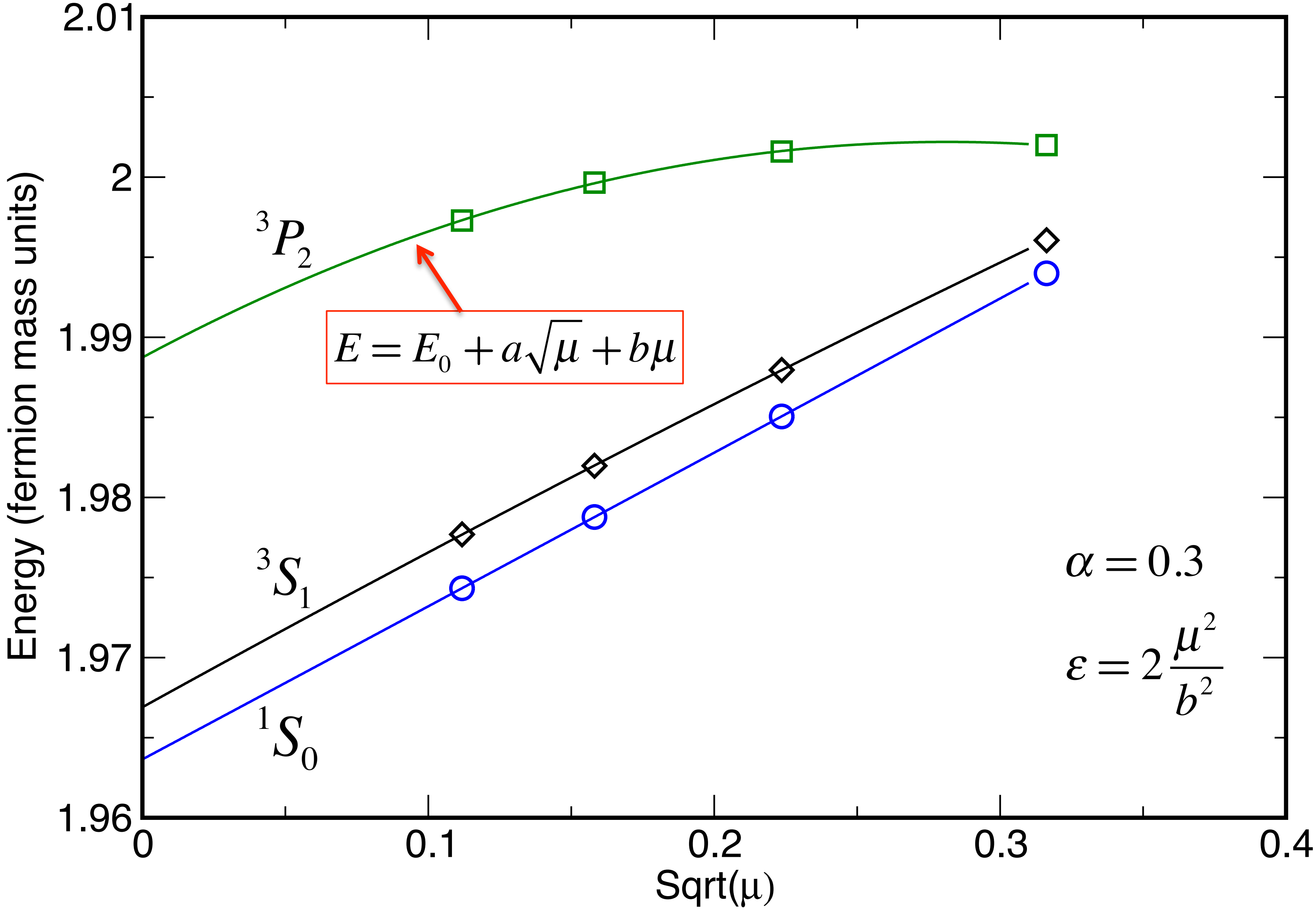


Ground State Energy (fermion mass units)
2.02
2.01
2
1.99
1.98
0
10
20
30
40
50
K
$\alpha = 0.3$
$\mu = 0.1 m_e$
$b = 0.5 m_e$
$\varepsilon = 2\frac{\mu^2}{b^2} = 0.08$
$E = a + be^{-c\sqrt{K}}$
$N_{max}=2$
$N_{max}=4$
$N_{max}=6$
$N_{max}=8$
$N_{max}=10$
$N_{max}=12$
$N_{max}=14$

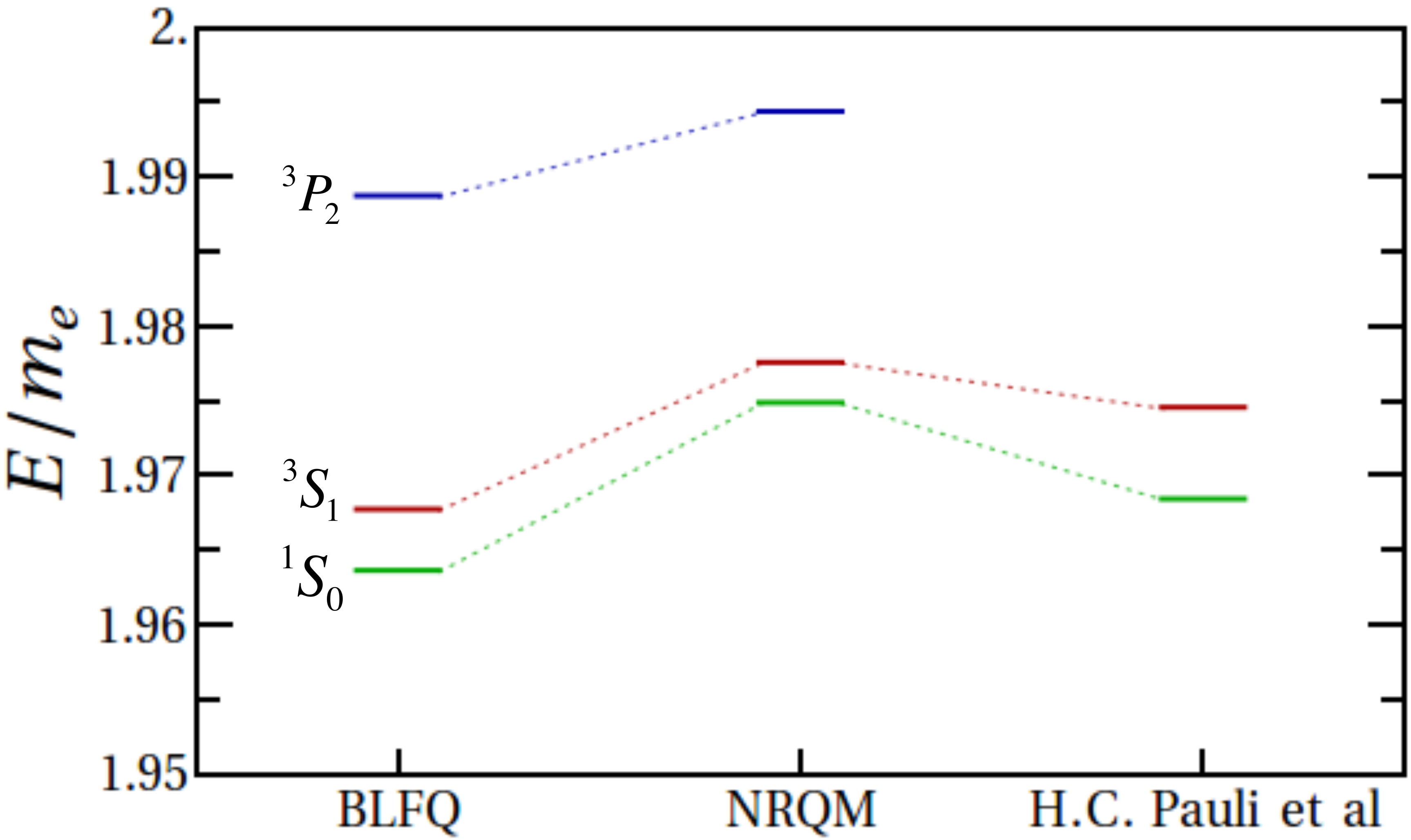


2.02
2.01
2
1.99
1.98
1.97
0
5
10
15
20
Ground State Energy (fermion mass units)
N max
$E = a' + b' e^{-c'\sqrt{N_{max}}}$
μ=0.0125
μ=0.025
μ=0.05
μ=0.1

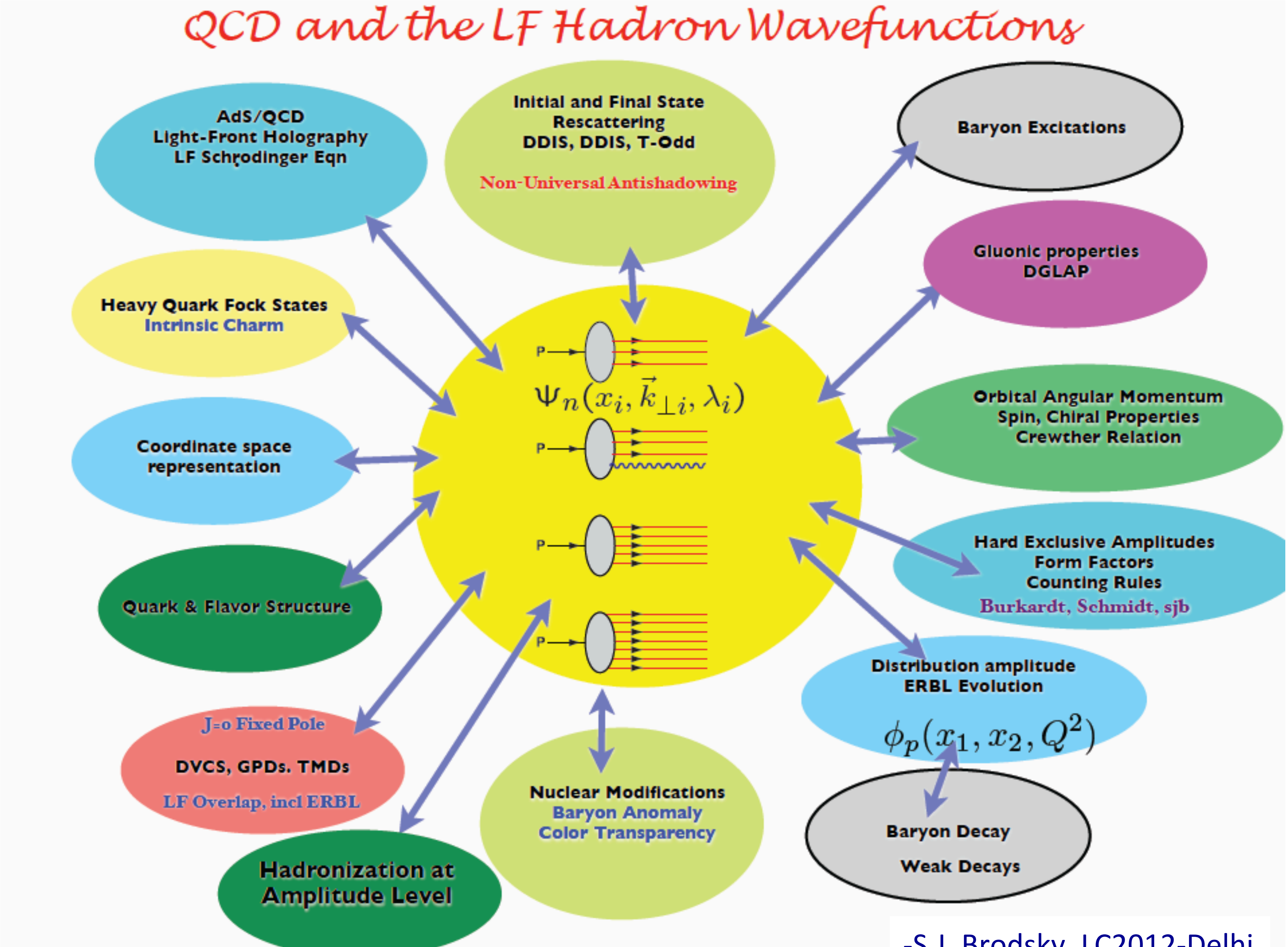
Positronium in the ε -> 0 limit after taking Nmax -> ∞ and K -> ∞



Positronium with $\alpha = 0.3$

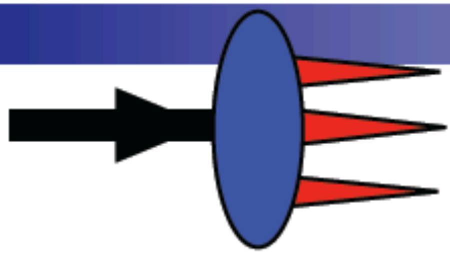


PRD 45 2968



-S.J. Brodsky, LC2012-Delhi

- *Light Front Wavefunctions:*



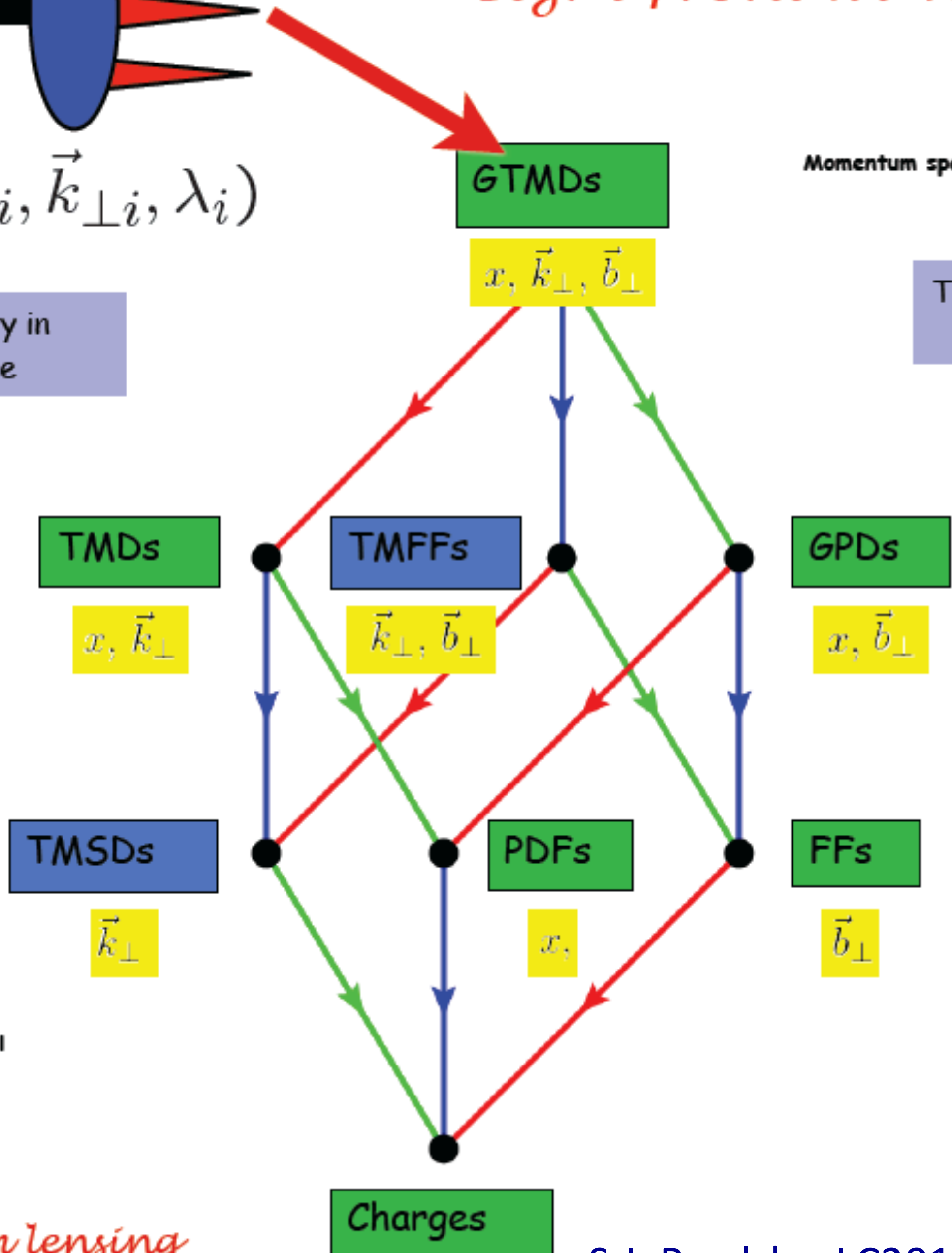
$$\Psi_n(x_i, \vec{k}_{\perp i}, \lambda_i)$$

Momentum space $\vec{k}_\perp \leftrightarrow \vec{z}_\perp$ Position space

$\vec{\Delta}_\perp \leftrightarrow \vec{b}_\perp$

Transverse density in momentum space

Transverse density in position space

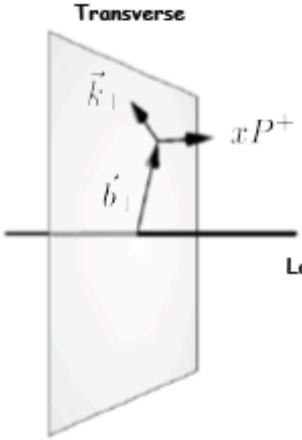


Lorce

$\int d^2 b_\perp$

$\int dx$

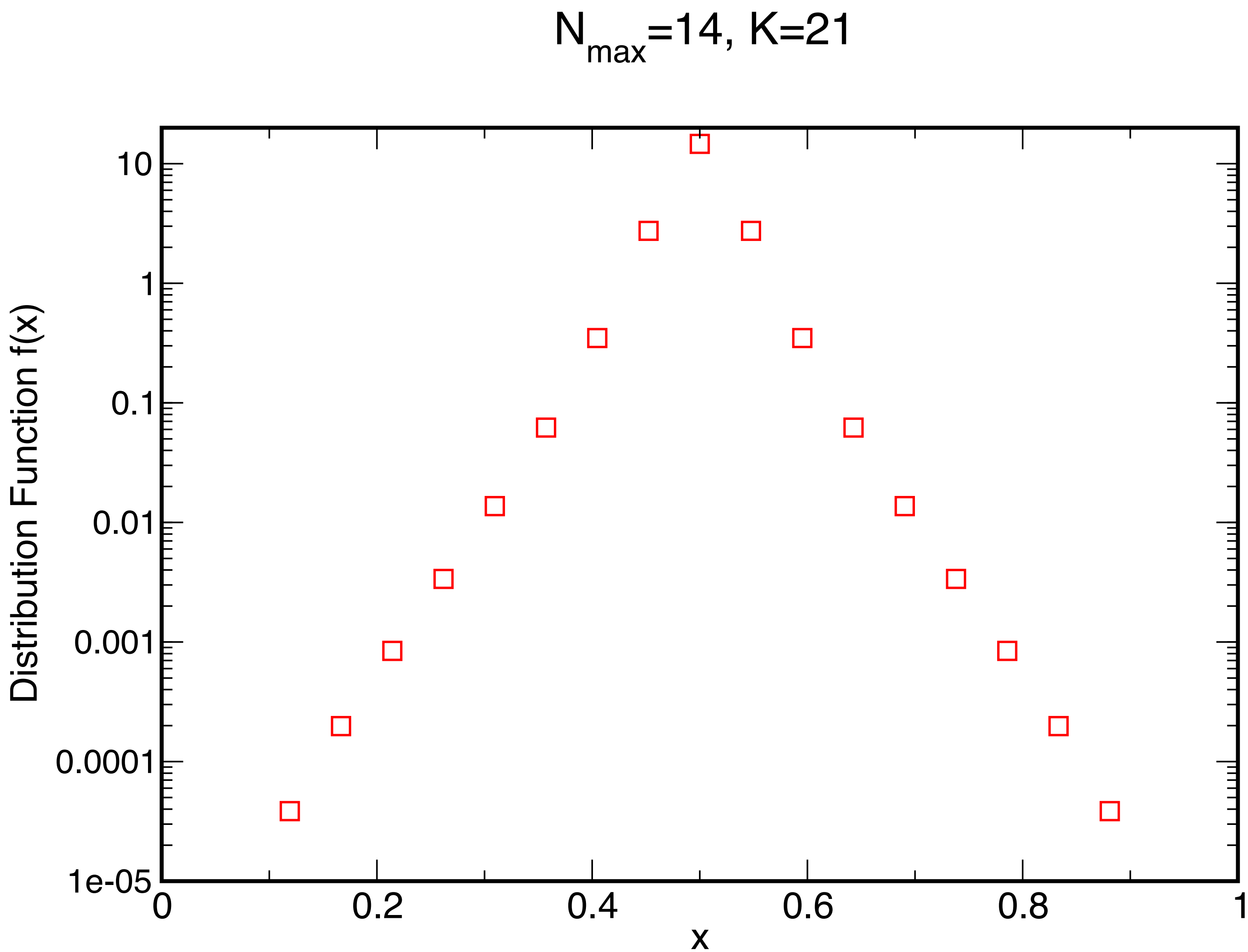
$\int d^2 k_\perp$



Sivers, T-odd from lensing

-S.J. Brodsky, LC2012-Delhi

Ground state Electron distribution function



Matrix Dimension = 10,584 & CPU hours = 737

Conclusions and Outlook

- BLFQ/tBLFQ are practical approaches to light-front QFT
- As for LGT, they are computationally intensive
- Massive amounts of computer resources are available
- Next up: mesons and baryons

Thank you

I welcome your questions!