Model of the AdS/QFT duality

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Abstract

Quantum field theory in the front form of dynamics [Dirac 1949] has been linked to the classical field theory in the 5-dimensional anti-deSitter (AdS) gravitational background by Brodsky and Teramond [Brodsky and Teramond 2008; Teramond and Brodsky 2009] in terms of the formula for a form factor of a hadron. We shall discuss the corresponding equations of motion and formulas for form factors in a model framework in which all these equations are analogs of the Ehrenfest equations of quantum mechanics. The required expectation values are obtained by integrating over relative motion variables and summing over quantum numbers of spectators. The AdS modes dual to the incoming and outgoing hadrons thus appear to be the parton Ehrenfest functions for the constituents that are struck by external probes.

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Introduction I

- Representation of physical states in the FF Fock space

$$\left|\mathrm{Hadron}: P^+, P^\perp\right\rangle = \sum_n \int [\mathbf{x},\mathbf{p}^\perp]\ \psi_P^{(n)}(\mathbf{x},\mathbf{p}^\perp;\lambda) \times \left|n : \mathbf{x}P^+, \mathbf{p}^\perp;\lambda\right\rangle .$$

- We shall omit all except momentum quantum numbers.
- In the proton: $|uud\rangle, |uudg\rangle, |uudq\bar{q}\rangle, |uudgg\rangle, \ldots$
- $\psi_P^{(n)}$ - wave functions of n-th Fock sector.
- λ - renormalization group (RG) scale.
- This state satisfies the FF Schrödginger eigenvalue equation.

Introduction II

- The Ehrenfest equation

$$\left[\vec{k}^{\,2} + \mathcal{M}^2 + U_{\text{eff}}\right] \psi(\vec{k}\,) = M^2 \psi(\vec{k}\,)\,,$$

- is another description of a hadron (Newton $\leftarrow$ Schrödinger),
- $\psi(\vec{k}\,)$ - the Ehrenfest function.
- $\psi(\vec{k}\,)$ describes the motion of an averaged active parton with respect to spectators.
- The Ehrenfest equation does not depend on λ (quantum RG parameter).
- Our point: (Newton $\xleftrightarrow[\text{Ehrenfest}]{}$ Schrödinger) $\Leftrightarrow$ (AdS/QFT duality)

Outline of the reasoning

The Ehrenfest equation I

$$\left|\mathrm{Hadron}\!: P^+, P^\perp\right\rangle = \sum_n \int [\mathbf{x}, \mathbf{p}^\perp]\ \psi_P^{(n)}(\mathbf{x}, \mathbf{p}^\perp; \lambda)\ \left|n : \mathbf{x}P^+, \mathbf{p}^\perp; \lambda\right\rangle$$

- The hadron state FF wave functions depend on the ratios of longitudinal momenta of constituents to the hadron longitudinal momentum, $\mathbf{x} = (x_i)_{i=1,\ldots,n}$, $x_i = p_i^+/P^+$,
- and transverse momenta of the constituents, $\mathbf{p}^\perp = (p_i^\perp)_{i=1,\ldots,n}$.

$$\int [\mathbf{p}^\perp] = \prod_i \int \frac{d^2 p_i^\perp}{(2\pi)^2} \qquad \text{and} \qquad \int [\mathbf{x}] = \prod_i \int \frac{dp_i^+}{(2\pi) 2p_i^+}\,.$$

The Ehrenfest equation II

- Our model of a hadron state is normalized

$$\big\langle \mathrm{Hadron}\colon P'^{+}, P'^{\perp} \big| \mathrm{Hadron}\colon P^{+}, P^{\perp} \big\rangle \\ = 2P^{+}(2\pi)^3 \delta(P'^{+} - P^{+})\, \delta^{(2)}(P'^{\perp} - P^{\perp})\,,$$

- and the relative momenta $\mathbf{k}^{\perp} = (k_i^{\perp})_{i=1,\dots,n}$ and $k_i^{\perp} = p_i^{\perp} - x_i P^{\perp}$ are separated from the total hadron momentum by writing

$$\psi_P^{(n)}(\mathbf{x}, \mathbf{p}^{\perp}; \lambda) = 2(2\pi)^3 \delta\Big(\sum_{i=1}^{n} x_i - 1\Big) \delta^{(2)}\Big(P^{\perp} - \sum_{i=1}^{n} p_i^{\perp}\Big) \\ \times \psi^{(n)}(\mathbf{x}, \mathbf{k}^{\perp}; \lambda)\,,$$

- $\sum_{i=1}^{n} k_i^{\perp} = 0$.

The Ehrenfest equation III

- The FF Hamiltonian $\hat{P}^-$ in a QFT determines the structure of a composite system through the eigenvalue equation in which the eigenvalue is expressible in terms of the kinematical momenta, P^+ and $P^\perp$, and mass squared, M^2, of the system

$$\hat{P}^- \left|\mathrm{Hadron}: P^+, P^\perp\right\rangle = \frac{M^2 + (P^\perp)^2}{P^+} \left|\mathrm{Hadron}: P^+, P^\perp\right\rangle .$$

- **In every sector, we will distinguish one constituent and we will focus on description of its motion with respect to other constituents.**
- The selected constituent will be called active, and it will be described using variables without subscripts.
- The other constituents will be called spectators or a core, depending on the context.

The Ehrenfest equation IV

- We can write the expectation value of $P^+\hat{P}^- - (P^\perp)^2$ in a hadron in the form

$$M^2 = \int \frac{d^2k^\perp\, dx}{2(2\pi)^3 x(1-x)} \sum_n \int [\mathbf{x}, \boldsymbol{\kappa}^\perp]$$
$$\times\ 2(2\pi)^3 \delta\Big(\sum_{j=1}^{n-1} x_j - 1\Big)\delta^{(2)}\Big(\sum_{j=1}^{n-1}\kappa_j^\perp\Big)$$
$$\times\ \psi_{k,x}^{\dagger(n)}\left[\frac{(k^\perp)^2}{x(1-x)} + \frac{m^2}{x} + \frac{\hat{\mathbf{m}}^2}{1-x}\right]\psi_{k,x}^{(n)}$$
$$+\,(\text{interactions})\,,$$

- The functions $\psi_{k,x}^{(n)}$ depend on x and $k^\perp$ separately from the core internal motion variables $\mathbf{x}$ and $\boldsymbol{\kappa}^\perp$ (actually, $n-1$ 3-dim variables).

The Ehrenfest equation V

- $\psi_{k,x}^{(n)}$ are eigenfunctions of the free invariant mass-squared operator $\hat{\mathbf{m}}^2$ for the spectators, with eigenvalue

$$\mathbf{m}_{n-1}^2 = \sum_{j=1}^{n-1} \frac{(\kappa_j^\perp)^2 + m^2}{x_j} .$$

- Since

$$\frac{m^2}{x} + \frac{\hat{\mathbf{m}}^2}{1-x} = \frac{[\hat{\mathbf{m}}\, x - m\,(1-x)]^2}{x(1-x)} + (m + \hat{\mathbf{m}})^2 ,$$

- we introduce the third component of relative momentum of the active constituent with respect to the core by writing

$$k^3 = \hat{\mathbf{m}}\, x - m\,(1-x) .$$

The Ehrenfest equation VI

- We now introduce

$$\vec{k} = (k_x, k_y, k_z) = \frac{(k^\perp, k^3)}{\sqrt{x(1-x)}}\,.$$

- Consequently, the Schrödinger expectation value equation becomes our Ehrenfest expectation value equation.
- Their free parts match each other according to the correspondence QM-CM formula

$$\left\langle \int d^3k\ \psi^{\dagger(n)}_{\vec{k}} \left[\vec{k}^{\,2} + (m + \hat{\mathbf{m}})^2\right] \psi^{(n)}_{\vec{k}} \right\rangle = \int d^3k\ \psi(\vec{k}\,)^\dagger \left[\vec{k}^{\,2} + \mathcal{M}^2\right] \psi(\vec{k}\,)\,,$$

- $\psi(\vec{k}\,)$ is our Ehrenfest function.

The Ehrenfest equation VII

- The averaging that is denoted by brackets $\langle\ \rangle$ satisfies conditions

$$\varkappa^2 = \left\langle \varkappa_n^2 \right\rangle \,,$$
$$\mathcal{M}^2 = \left\langle (m + \mathbf{m}_{n-1})^2 \right\rangle \,,$$
$$U_{\text{eff}} = \langle \text{potential} \rangle \,.$$

- The $\varkappa^2$ corresponds to the half-width of the Ehrenfest function $\psi(\vec{k}\,)$, whereas $\varkappa_n^2$ corresponds to the half-width of $\psi_{\vec{k}}^{(n)}$.
- $\mathcal{M}^2$ is the expectation value of the combined mass squared of the active constituent and core, averaged over spectators' relative dynamics in the core and the sectors that form the hadron of mass M.
- The mass of a hadron M differs form $\mathcal{M}$ by the biding energy caused by the effective potential U_{eff}.

The Ehrenfest equation VIII

- Variation of the Ehrenfest expectation value equation with respect to $\psi(\vec{k}\,)$, keeping the norm fixed (total charge), yields the Ehrenfest equation,

$$\left[\vec{k}^{\,2} + \mathcal{M}^2 + U_{\text{eff}}\right]\psi(\vec{k}\,) = M^2\psi(\vec{k}\,)\,.$$

Form factors I

- Now we repeat our procedure for the form factor of a hadron, $F(q^2)$, that is another observable.
- We do this, following Brodsky and Teramond, using our Ehrenfest function.
- We suggest that the Brodsky-Teramond holographic density corresponds to the modulus squared of our Ehrenfest function.
- The form factor defined in terms of a matrix element of the current $\hat{J}^+(x=0)$ with $q^+=0$ is

$$\begin{aligned}&\left\langle \mathrm{Hadron}: P^+, P^\perp + q^\perp \middle| \hat{J}^+(0) \middle| \mathrm{Hadron}: P^+, P^\perp \right\rangle \\ &\quad = Q_{\mathrm{Hadron}}\; 2P^+\; F(q^2)\,,\end{aligned}$$

- where Q_{Hadron} denotes the relevant charge of the hadron and $F(0)=1$.

Form factors II

- The form factor formula reads [Drell and Yan 1970; West 1970]

$$F(q^2) = \sum_n \int [\mathbf{x},\mathbf{p}^\perp] \sum_{j=1}^{n} e_j$$
$$\times\, 2(2\pi)^3 \delta\Big(\sum_{i=1}^{n} x_i - 1\Big)\delta^{(2)}\Big(\sum_{i=1}^{n} p_i^\perp - P^\perp\Big)$$
$$\times\, \psi^{\dagger(n)}(\mathbf{p_j}^\perp - \mathbf{x}P'^\perp, \mathbf{x}; \lambda)\, \psi^{(n)}(\mathbf{p}^\perp - \mathbf{x}P^\perp, \mathbf{x}; \lambda)\,,$$

- where e_j is the fraction of hadron charge carried by the active constituent.
- $\mathbf{p_j}^\perp = (p_i^\perp + \delta_{ij}\, q^\perp)_{i=1,\ldots,n}$, and $P'^\perp - P^\perp = q^\perp$.
- For simplicity, we assume that $\psi^{(n)}$ is a symmetric function of momenta.
- Then it is easy to see that one can choose a value of j, for instance n, and carry out the summation over active constituents, which produces $\sum_j e_j = 1$ in each Fock sector.

Form factors III

- So, the Fourier transform in $k^\perp$ gives

$$
\begin{aligned}
F(q^2) = \int \frac{d^2\eta^\perp\, dx}{4\pi x(1-x)} \sum_n \int [\mathbf{x}, \boldsymbol{\kappa}^\perp] \\
\times\ 2(2\pi)^3 \delta\Big(\sum_j x_j - 1\Big)\delta^{(2)}\Big(\sum_j \kappa_j^\perp\Big) \\
\times\ e^{i(1-x)\eta^\perp q^\perp}\ \left|\tilde{\psi}^{(n)}_{\eta,x}\big(\boldsymbol{\kappa}^\perp, \mathbf{x}; \lambda\big)\right|^2 ,
\end{aligned}
$$

- where $\left|\tilde{\psi}^{(n)}_{\eta,x}\right|^2$ is the Brodsky-Teramond density, which can be alternatively described in terms of our Ehrenfest function.

Model of AdS/QFT duality I

We now describe a simple model that illustrates the Ehrenfest interpretation of the Brodsky-Teramond holography:

- We assume that each constituent is attracted by some force to the core formed by other constituents.
- More or less like an electron is attracted to the center of positive charge distribution in the Thomson model of an atom, except that we have to average over various numbers of constituents.
- Quite generally, one can assume that an active constituent is attracted to a minimum of the effective potential that by necessity is a quadratic function of the distance between the active constituent and the core center.

Model of AdS/QFT duality II

- If an absolute transverse position of i-th constituent is denoted by $r_i^\perp$, then the position of a hadron is

$$R^\perp = \sum_{i=1}^{n} x_i r_i^\perp \,,$$

- and the position of *the center of mass of the core* is

$$R_j^\perp = \frac{\sum_{i\neq j} x_i r_i^\perp}{\sum_{i\neq j} x_i}\,.$$

- One denotes by $\eta_i^\perp$ the relative distance between i-th constituent and the position of the hadron,

$$\begin{aligned}\eta_i^\perp &= r_i^\perp - R^\perp \\ &= (1-x_i)(r_i^\perp - R_i^\perp)\,.\end{aligned}$$

- Relative momenta $\mathbf{k}^\perp = (k_i^\perp)_{i,\ldots,n}$ are canonically conjugated to the $\boldsymbol{\eta}^\perp = (\eta_i^\perp)_{i,\ldots,n}$.

Model of AdS/QFT duality III

- In position variables, the FF wave function of or model state is

$$\tilde{\psi}^{(n)}(\boldsymbol{\eta}^{\perp}, \mathbf{x}; \lambda) = \varkappa_n^{\,n} \; \tilde{A}_n\Big(\lambda/\Lambda_{QFT}\Big) \\ \times \exp\left\{ -\frac{1}{4} \sum_{i=1}^{n} \left[(\eta_i^{\perp})^2 \, x_i \, \varkappa_n^2 + \frac{m^2}{x_i} / \varkappa_n^2 \right] \right\}.$$

- On the other hand, in momentum space,

$$\psi^{(n)}(\mathbf{k}^{\perp}, \mathbf{x}; \lambda) = \frac{A_n\Big(\lambda/\Lambda_{QFT}\Big)}{\varkappa_n^{\,n}} \\ \times \exp\left\{ -\frac{1}{4} \sum_{i=1}^{n} \left[\frac{m^2 + (k_i^{\perp})^2}{x_i} \right] / \varkappa_n^2 \right\} .$$

- $\varkappa_n^2 = \varkappa_n^2(\lambda)$ are the widths of the Fock Gaussian wave functions.

Model of AdS/QFT duality IV

- The normalization condition gives

$$1 = \int \frac{d^2k^\perp\, dx}{2(2\pi)^3 x(1-x)} \sum_n \frac{|A_n|^2}{\varkappa_n^{2n}} \int [\mathbf{x},\boldsymbol{\kappa}^\perp]$$

$$\times\, 2(2\pi)^3 \delta\Big(\sum_{j=1}^{n-1} x_j - 1\Big) \delta^{(2)}\Big(\sum_{j=1}^{n-1} \kappa_j^\perp\Big)$$

$$\times \exp\left\{ -\frac{1}{2} \left[\frac{(k^\perp)^2}{x(1-x)} + \frac{m^2}{x} + \frac{\mathbf{m}_{n-1}^2}{1-x} \right] / \varkappa_n^2 \right\} .$$

- So instead of integration over all $\mathbf{x}$ and $\boldsymbol{\kappa}^\perp$ one can integrate over a single variable $\mathbf{m}^2$ together with density ρ_n,

$$\rho_n\left(\mathbf{m}^2\right) = \frac{1}{\varkappa_n^{2(n-1)}} \int [\mathbf{x},\boldsymbol{\kappa}^\perp]\, 2(2\pi)^3 \delta\Big(\sum_j x_j - 1\Big) \delta^{(2)}\Big(\sum_j \kappa_j^\perp\Big)$$

$$\times\, \delta\Big(\mathbf{m}^2 - \mathbf{m}_{n-1}^2(\mathbf{x}, \boldsymbol{\kappa}^\perp)\Big) .$$

Model of AdS/QFT duality V

- Finally, we are able to write explicitly the expectation value of a quantity X in our model state as

$$\langle X \rangle = \sum_n \int d\mathbf{m}^2 \;\rho_n(\mathbf{m}^2) \frac{\partial(k^1, k^2, x)}{\partial(k_x, k_y, k_z)} \left(\frac{\varkappa_n}{2}\right) \times |A_n|^2 \, e^{-(m+\mathbf{m})^2/4\varkappa_n^2} \, X \; e^{-(m+\mathbf{m})^2/4\varkappa_n^2} \,,$$

- where X can be any function of $\mathbf{m}$ or n.
- The Ehrenfest expectation value equation for the free part of the FF Hamiltonian, in this case mass squared, reads

$$\left\langle \int \frac{d^3k}{(2\pi)^3} \frac{e^{-\vec{k}^{\,2}/4\hat{\varkappa}^2}}{\sqrt{\hat{\varkappa}}^{\,3}} \left[\, \vec{k}^{\,2} + (m + \hat{\mathbf{m}})^2 \right] \frac{e^{-\vec{k}^{\,2}/4\hat{\varkappa}^2}}{\sqrt{\hat{\varkappa}}^{\,3}} \right\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-\vec{k}^{\,2}/4\varkappa^2}}{\sqrt{\varkappa}^{\,3}} \left[\vec{k}^{\,2} + \mathcal{M}^2\right] \frac{e^{-\vec{k}^{\,2}/4\varkappa^2}}{\sqrt{\varkappa}^{\,3}} \,.$$

Model of AdS/QFT duality VI

- Thus, we find that the Ehrenfest function (with proper normalization) is

$$\psi(\vec{k}\,) = \left(\frac{2\pi}{\varkappa^2}\right)^{3/4} e^{-\vec{k}^{\,2}/4\varkappa^2} \,,$$

- and it by necessity satisfies the Ehrenfest equation with a harmonic oscillator potential,

$$\hat{U}_{\text{eff}} = -\varkappa^4 \left(\frac{\partial}{\partial \vec{k}}\right)^2 \,.$$

Conclusion

- We have shown on the model example that the QFT Schrödginger equation can be quite generally expected to yield the Ehrenfest equation with a harmonic oscillator potential.
- Our usage of a factorized, i.e., Gaussian FF wave functions of perpendicular and longitudinal variables gives

$$M^2\ \tilde{\phi}(\zeta) = \Bigg[\mathcal{M}^2 - \left(\frac{\partial}{\partial\zeta}\right)^2 - \frac{1}{\zeta}\frac{\partial}{\partial\zeta} + \frac{1}{\zeta^2} l_z^2 + (2L+1)\,\varkappa^2 + \varkappa^4\zeta^2\Bigg]\tilde{\phi}(\zeta)\,,$$

- where ζ is the length of a position vector canonically conjugated to momentum (k_x, k_y) and l_z is angular momentum projection on z-axis.
- In this form, our Ehrenfest equation resembles the ones used in AdS/QFT duality calculations, where ζ is associated with the 5-th dimension z and modified metric with a soft-wall potential.

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