Scale and Lorentz transformations at the light front

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Outlook

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1. *Definitions and basic assumptions about the light front*
   - Definitions
   - Basic assumptions about the light front
2. *2-point Wightman function*
   - Lorentz transformation
   - Lorentz transformation at LF
   - Scale transformation
3. *Distributional solution*
   - Fourier transform
   - LF restriction of Weinberg's results
4. *Conclusions and prospects*

- light-cone (LC) coordinates

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}, \quad x^- = \frac{x^0 - x^3}{\sqrt{2}}, \quad \mathbf{x}_\perp^i = (x^1, x^2),\ i = 1, 2;$$

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- x^+ light-front time parameter

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- *a null plane field theory is dilatation invariant in the null plane even if it has a mass*

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- therefore one gets only the imaginary part of 2-WF

$$\Im\langle 0|\phi(0,\bar{x})\,\phi(0,\bar{y})|0\rangle = -\frac{1}{8}\,\mathrm{sgn}(x^- - y^-)\,\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp)$$

Outlook

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$$U(\Lambda)\,\phi(x)\,U^{-1}(\Lambda) = \phi(x') \approx \phi(x) + \frac{1}{2}\,\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x)$$

where $\Lambda_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu}$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$,

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- or in the LC coordinates

$$x^+\partial_i W_{(2)}(x^+,\bar{x}) + x^i\partial_- W_{(2)}(x^+,\bar{x}) = 0, \qquad \Leftarrow \qquad \boxed{J_{-i}}$$

$$x^-\partial_i W_{(2)}(x^+,\bar{x}) + x^i\partial_+ W_{(2)}(x^+,\bar{x}) = 0, \qquad \Leftarrow \qquad \boxed{J_{+i}}$$

$$x^+\partial_+ W_{(2)}(x^+,\bar{x}) - x^-\partial_- W_{(2)}(x^+,\bar{x}) = 0, \qquad \Leftarrow \qquad \boxed{J_{-+}}$$

$$\left(x^i\partial_j - x^j\partial_i\right) W_{(2)}(x^+,\bar{x}) = 0, \qquad \Leftarrow \qquad \boxed{J_{ij}}$$

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$$\lim_{x^+\to 0} x^-\partial_- W_{(2)}(x^+,\bar{x}) = \boxed{x^-\partial_- W_{(2)}(0,\bar{x}) = -\frac{1}{4\pi}\delta^2(\mathbf{x}_\perp)} \neq 0$$

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$$J_{+-} = x^+P^- - \int \mathrm{d}^3\bar{x}\, T^{++}x^-$$

is a kinematical generator, because it LF limit is

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- thus we need to use the complete J_{+-}

$$\lim_{x^+\to 0} x^+\partial_+ W_{(2)}(x^+,\bar{x}) = \lim_{x^+\to 0} x^-\partial_- W_{(2)}(x^+,\bar{x}) = -\frac{1}{4\pi}\delta^2(\mathbf{x}_\perp) \neq 0.$$

for $x^+ \sim 0$, $W_{(2)}(x^+,\bar{x})$ has a logarithmic dependence on x^+ and x^- along a light-like direction

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or equivalently

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- since $x^- \partial_- W_{(2)}(0,\bar{x}) = -\frac{1}{4\pi}\delta^2(\mathbf{x}_\perp)$ then we obtain

$$\partial_i \left[x^i W_{(2)}(0,\bar{x}) \right] = \frac{1}{2\pi}\delta^2(\mathbf{x}_\perp),$$

from distributions $\mathcal{S}'(\mathbb{R}^2)$ we have

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for $x^i W_{(2)}(0,\bar{x})$ and $\boxed{x^- \partial_- W_{(2)}(0,\bar{x}) = -\frac{1}{4\pi}\delta^2(\mathbf{x}_\perp)}$ one checks the relation

$$\boxed{\Delta_\perp[x^i W_{(2)}(0,\bar{x})] + 2x^- \partial_- \partial_i W_{(2)}(0,\bar{x}) = 0}$$

Outlook

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- the Fourier transform exists

$$\int \frac{\mathrm{d}^2\mathbf{x}_\perp}{(2\pi)^2} e^{+\mathrm{i}\mathbf{k}_\perp\cdot\mathbf{x}_\perp}\frac{x_\perp^i}{x_\perp^2}\ln x_\perp = -\frac{\mathrm{i}}{2\pi}\frac{k_\perp^i}{k_\perp^2}\left[\ln\left(\frac{k_\perp}{2}\right) + \gamma_E\right]$$

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and then by rearranging terms

$$\Delta_{\perp}[x^{i}W_{(2)}(x)]+2x^{+}\partial_{+}\partial_{i}W_{(2)}(x)+x^{i}\left[2\partial_{-}\partial_{+}-\Delta_{\perp}\right]W_{(2)}(x)=0$$

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thus at LF one finds

$$\boxed{\Delta_\perp[x^i W_{(2)}(0,\bar{x})] + 2x^-\partial_-\partial_i W_{(2)}(0,\bar{x})} + x^i \lim_{x^+\to 0}\left[2\partial_-\partial_+ - \Delta_\perp\right] W_{(2)}(x) = 0$$

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$$\boxed{\Delta_\perp[x^i W_{(2)}(0,\bar{x})] + 2x^-\partial_-\partial_i W_{(2)}(0,\bar{x})} + x^i \lim_{x^+\to 0}\left[2\partial_-\partial_+ - \Delta_\perp\right] W_{(2)}(x) = 0$$

thus

$$x^i \lim_{x^+\to 0} \Box W_{(2)}(x) = 0 \qquad \Box = 2\partial_-\partial_+ - \Delta_\perp$$

Starting from the Lorentz transformation (with J_{-i})

$$2\left(x^{+}\partial_{i}+x^{i}\partial_{-}\right)W_{(2)}(x)=0$$

by applying ∂_{+} one finds

$$2\left(x^{+}\partial_{+}\partial_{i}+\partial_{i}+x^{i}\partial_{-}\partial_{+}\right)W_{(2)}(x)=0$$

and then by rearranging terms

$$\Delta_{\perp}[x^{i}W_{(2)}(x)]+2x^{+}\partial_{+}\partial_{i}W_{(2)}(x)+x^{i}\left[2\partial_{-}\partial_{+}-\Delta_{\perp}\right]W_{(2)}(x)=0$$

thus at LF one finds

$$\boxed{\Delta_{\perp}[x^{i}W_{(2)}(0,\bar{x})]+2x^{-}\partial_{-}\partial_{i}W_{(2)}(0,\bar{x})}+x^{i}\lim_{x^{+}\to 0}\left[2\partial_{-}\partial_{+}-\Delta_{\perp}\right]W_{(2)}(x)=0$$

thus

$$x^{i}\lim_{x^{+}\to 0}\Box W_{(2)}(x)=0 \qquad \Box=2\partial_{-}\partial_{+}-\Delta_{\perp}$$

it is LF restriction of Weinberg's equation

$$x^{2}\Box G(x)=0, \qquad G(x)=W_{(2)}(x)$$

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Minimal fields of canonical dimensionality are free

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It is shown that in a scale-invariant relativistic field theory, any field ψ_n belonging to the $(j, 0)$ or $(0, j)$ representations of the Lorentz group and with dimensionality $d = j + 1$ is a free field. For other field types there is no value of the dimensionality that guarantees that the field is free. Conformal invariance is not used in the proof of these results, but it gives them a special interest; as already known and as shown here in the appendix, the only fields in a conformal field theory that can describe massless particles belong to the $(j, 0)$ or $(0, j)$ representations of the Lorentz group and have dimensionality $d = j + 1$. Hence in conformal field theories massless particles are free.

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This note will show that in a scale-invariant relativistic field theory, any fields that belong to the minimal $2j + 1$-component $(j, 0)$ or $(0, j)$ representations of the Lorentz group (where j is an integer or half-integer) and have canonical dimensionality $d = j + 1$ are necessarily free fields. This conclusion is already known for $j = 0$ [1]; here it is extended to all spins. Although conformal invariance is not used here, this result gains interest from the fact [2] that in conformal theories the only primary fields that can describe massless particles belong to the $(j, 0)$ and $(0, j)$ representations of the Lorentz group and have canonical dimensionality. An elementary proof of this theorem is given in the Appendix. It follows that, according to the main result of the present paper, massless particles in a conformally invariant field theory must be free particles.

The point of this exercise is that by elementary commutations of derivatives and coordinates, one can derive the identity

$$\mathcal{L}^{\rho\sigma}\mathcal{L}_{\rho\sigma} = -2z^2\Box + 2S^2 - 4S, \tag{5}$$

where $\Box \equiv \partial^2/\partial z^\rho \partial z_\rho$ is the usual d'Alembertian, and S is the scale transformation operator

$$S \equiv -z^\rho \frac{\partial}{\partial z^\rho}. \tag{6}$$

[This is analogous to the identity in three dimensions that can be used to show that the Laplacian of the spherical polynomial $r^\ell Y_\ell^m(\theta, \phi)$ vanishes.] We will use Eqs. (4)–(6) to show that if ψ belongs to the $(j, 0)$ or $(0, j)$ representadimensionality. An elementary proof of this theorem is given in the Appendix. It follows that, according to the main result of the present paper, massless particles in a conformally invariant field theory must be free particles.

To begin, consider a field $\psi_n(x)$ belonging to any representation of the Lorentz group. Poincaré invariance tells us that

$$\mathcal{L}_{\rho\sigma}G_{nm}(z) = -\sum_l [\mathcal{J}_{\rho\sigma}]_{nl}G_{lm}(z) + \sum_l G_{nl}(z)[\mathcal{J}^\dagger_{\rho\sigma}]_{lm}, \tag{1}$$

where G is the vacuum expectation value

$$G_{nm}(x-y) \equiv \langle 0|\psi_n(x)\psi^\dagger_m(y)|0\rangle, \tag{2}$$

$\mathcal{L}_{\rho\sigma}$ are the differential operators

$$\mathcal{L}_{\rho\sigma} \equiv -iz^\rho \frac{\partial}{\partial z_\sigma} + iz^\sigma \frac{\partial}{\partial z_\rho}, \tag{3}$$

and $[\mathcal{J}_{\rho\sigma}]_{nm}$ are the matrices representing the generators of the Lorentz group in the representation furnished by the field $\psi(x)$. Iteration of Eq. (1) gives (suppressing matrix indices)

$$\mathcal{L}^{\rho\sigma}\mathcal{L}_{\rho\sigma}G(z) = \mathcal{J}^{\rho\sigma}\mathcal{J}_{\rho\sigma}G + G\mathcal{J}^{\dagger\,\rho\sigma}\mathcal{J}^\dagger_{\rho\sigma} - 2\mathcal{J}^{\rho\sigma}G\mathcal{J}^\dagger_{\rho\sigma}. \tag{4}$$

[This is analogous to the identity in three dimensions that can be used to show that the Laplacian of the spherical polynomial $r^\ell Y^m_\ell(\theta,\phi)$ vanishes.] We will use Eqs. (4)–(6) to show that if ψ belongs to the $(j,0)$ or $(0,j)$ representations of the Lorentz group and has canonical dimensionality then $\Box\psi = 0$.

If $\psi(x)$ belongs to the $(j,0)$ representation of the Lorentz group, then

$$\mathcal{J}_{ij} = \epsilon_{ijk}\mathcal{J}_k, \qquad \mathcal{J}_{i0} = -i\mathcal{J}_i, \tag{7}$$

where $\mathcal{J}_i$ are the Hermitian matrices representing the generators of the rotation group in its spin j representation. It follows that

$$\begin{gathered} \frac{1}{2}\mathcal{J}^{\rho\sigma}\mathcal{J}_{\rho\sigma}G = 2\mathcal{J}_i\mathcal{J}_iG = 2j(j+1)G \\ \frac{1}{2}G\mathcal{J}^{\dagger\rho\sigma}\mathcal{J}^\dagger_{\rho\sigma} = 2G\mathcal{J}^\dagger_i\mathcal{J}^\dagger_i = 2j(j+1)G \\ \mathcal{J}^{\rho\sigma}G\mathcal{J}^\dagger_{\rho\sigma} = 0. \end{gathered} \tag{8}$$

Also, if ψ has dimensionality d (counting powers of momentum) then in a scale-invariant theory

$$SG(z) = 2dG(z). \tag{9}$$

So for these fields, Eq. (4) reads

$$-2z^2\Box G(z) + 8d^2G(z) - 8dG(z) = 8j(j+1)G(z), \tag{10}$$

and in particular, for $d = j+1$,

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there is no scale symmetry at LF for a free massive scalar field

Conclusions and prospects

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- LF analysis allows to calculate the Wightman function without introducing explicitly a Fock space for quantum field operators

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Thank you for your kind attention