Wavelet methods in field theory

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What are wavelets?

- They are orthonormal basis functions that are used in data compression algorithms.
- JPEG digital images are tables of expansion coefficients in a wavelet basis.
- FBI fingerprint files are stored as expansion coefficients in a wavelet basis.
- They are fractal valued.

Outline

- Summary of useful properties (for field theories)
- Construction of the basis
- Properties of the basis
- Multi-scale fields
- Renormalization group
- Symmetries and partitions of unity
- Truncations and approximations

Useful properties of wavelet bases (for field theory)

- **They are a basis for $L^2(\mathbb{R})$.**
  - **Fields can be expressed exactly as linear combinations of discrete field operators and known coefficient functions.**
- **They have compact support.**
  - **The discrete fields form a local algebra of operators.**
  - **Natural separation of scales**
  - **Natural resolution and volume cutoffs.**

- **The basis functions are related to fixed points of a renormalization group equation.**
  - **The basis is natural for formulating renormalization group transformations.**
  - **The basis naturally separates physics associated with different scales.**
- **The basis contains partitions of unity on all scales.**
  - **The partitions of unity are natural for studying how truncations violate Poincaré invariance.**
- **The basis functions are differentiable; finite difference approximations can be replaced by derivatives.**
- **Quadratures needed for field theory applications can be computed exactly using renormalization group methods.**

Basis construction

operators

$$\underbrace{(Df)(x) = \sqrt{2}f(2x)}_{\textbf{scale change}} \qquad \underbrace{(Tf)(x) = f(x-1)}_{\textbf{translation}}.$$

scaling equation

$$s(x) = D(\sum_{l=0}^{2K-1} h_l T^l s(x)) \qquad \int s(x)dx = 1$$

h_l **fixed constants**

$s(x)$ **:= scaling function**

Renormalization group transformation

$$f_n(x) = \underbrace{D\,(\underbrace{\sum_{l=0}^{2K-1} h_l T^l f_{n-1}(x)}_{\textbf{block average}})}_{\textbf{rescale}}$$

$s(x)$ is a fixed point of this Renormalization group transformation!

The h_n are solutions to (for each K)

$$\sum_{n=0}^{2K-1} h_n = \sqrt{2} \qquad \textbf{existence}$$

$$\sum_{n=0}^{2K-1} h_n h_{n-2m} = \delta_{m0} \qquad \left(\int s(x-n)s(x) = \delta_{n0}\right)$$

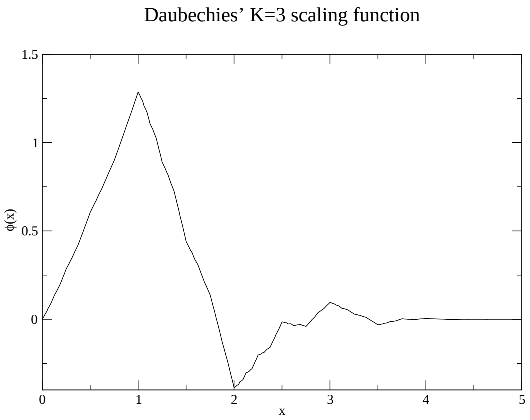
$$\sum_{n=0}^{2K-1} n^m (-l)^n h_{2K-1-n} = 0 \qquad m < K$$

$$\left(x^m = \sum_n c_n(m) s(x-n) \qquad m < K\right)$$

Equations fix h_n up to reflection, $h_n \rightarrow h'_n = h_{2K-1-n}$

Solutions: Daubechies' scaling coefficients, $K = 1, 2, 3$

h_l	**K=1**	**K=2**	**K=3**
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$


Daubechies' K=3 scaling function
φ(x)
x
1.5
1
0.5
0
0
1
2
3
4
5

Properties of the scaling function $s(x)$

1. **Reality**

$$s(x) = s^*(x)$$

2. **Partition of unity**

$$1 = \sum_{n=-\infty}^{\infty} s(x-n) = \sum_{n=-\infty}^{\infty} (T^n s)(x)$$

3. **Compact support**

$$\textbf{support}[s(x)] = [0, 2K-1]$$

5. **Differentiability (K>2)**

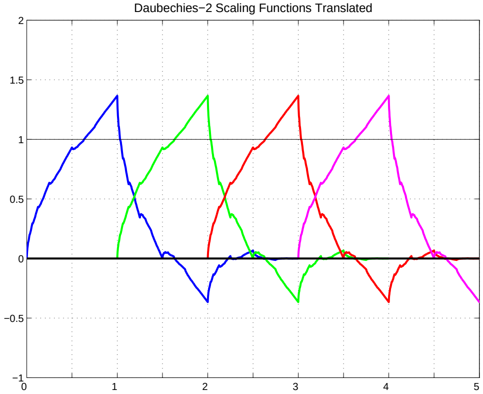
$$\frac{ds(x)}{dx} \qquad \textbf{exists} \qquad C^1(\mathbb{R}) \qquad \textbf{for} \qquad K=3$$

6. **Orthonormality**

$$(T^m s, T^n s) = \delta_{mn}$$

7. **Pointwise low-degree polynomial representation**

Partition of unity



Resolution $1/2^k$ spaces

$$s_n^k(x) := (D^k T^n s)(x) = \sqrt{2^k} s(2^k(x - n/2^k))$$

Resolution $1/2^k$ subspace $\mathcal{S}_k$

$$\mathcal{S}_k := \{f(x) | f(x) = \sum_{n=-\infty}^{\infty} c_n s_n^k(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}$$

The scaling equation implies

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{S}_{k+1} \supset \mathcal{S}_k \supset \mathcal{S}_{k-1} \supset \cdots$$

Properties of $s_n^k(x)$

1. **Reality:**

$$s_n^k(x) = s_n^{k*}(x)$$

2. **Partition of unity:**

$$\frac{1}{\sqrt{2^k}} \sum_{n=-\infty}^{\infty} s_n^k(x) = \sum_{n=-\infty}^{\infty} s(2^k x - n) = 1$$

3. **Compact support:**

$$\textbf{support}[s_n^k(x)] = [\frac{n}{2^k}, \frac{n + 2K - 1}{2^k}]$$

4. **Differentiability (continuous for $K \geq 3$):**

$$\frac{ds_n^k(x)}{dx} = 2^k D^k T^n \frac{ds}{dx}$$

$$\frac{d}{dx} D = 2D \frac{d}{dx} \qquad \frac{d}{dx} T = T \frac{d}{dx}$$

5. **Orthonormality:**

$$(s^k_m, s^k_n) = \delta_{mn}$$

6. **Approximation:**

$$\lim_{k\to\infty} \mathcal{S}_k = L^2(\mathbb{R})$$

7. **Normalization (scale fixing):**

$$\int s^k_n(x)dx = \frac{1}{\sqrt{2^k}}$$

8. **Pointwise low-degree polynomial**

$$x^m = \sum_n c^k_n(m) s^k_n(x) \qquad m < K$$

Multi-scale decomposition of $L^2(\mathbb{R})$

$$m > n \quad \Rightarrow \quad \mathcal{S}_m \supset \mathcal{S}_n$$

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{S}_{n+1} \supset \mathcal{S}_n \supset \mathcal{S}_{n-1} \supset \cdots \supset \emptyset$$

$$\mathcal{S}_{n+1} = \mathcal{S}_n \oplus \mathcal{W}_n$$

$$\Downarrow$$

$$\mathcal{S}_n = \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \cdots \oplus \mathcal{W}_{n-m} \oplus \mathcal{S}_{n-m}$$

$$\lim_{n\to\infty} \mathcal{S}_n = L^2(\mathbb{R})$$

$$L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{W}_n = \mathcal{S}_m \oplus \left(\bigoplus_{n=m}^{\infty} \mathcal{W}_n \right)$$

Wavelets

$\mathcal{W}_n$ **are wavelet spaces**

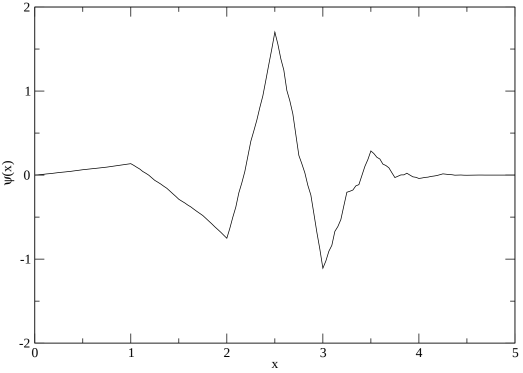
$$w(x) := D(\sum_{l=0}^{2K-1} (-)^l h_{2K-l-1} T^l s(x)) = D(\sum_{l=0}^{2K-1} g_l T^l s(x))$$

$w(x)$ **is called the "Mother" wavelet**

$$w_l^n(x) := D^n T^l w(x)$$

$\{w_l^n\}_{l=-\infty}^{\infty}$ **orthonormal basis for** $\mathcal{W}_n$

Daubechies' $K = 3$ mother wavelet



Properties of wavelets

- **support** $[w_m^k(x)]$ **= support** $[s_m^k(x)]$

- $\int w_m^k(x) x^n dx = 0 \qquad 0 \le n \le K-1$

- $\int w_m^k(x) w_n^l(x) dx = \delta_{kl}\delta_{mn}$

- $\int s_m^k(x) w_n^{k+l}(x) dx = 0 \qquad l \ge 0$

Multi-resolution basis

$$\{s_n^k(x)\}_{n=-\infty}^{\infty} \cup \{w_n^{k+l}(x)\}_{n=-\infty}^{\infty}{}_{l=0}^{\infty}$$

All functions have compact support.

The basis is orthonormal.

Multidimensional basis functions are products of one-dimensional basis functions.

Change of scale (orthogonal transformation relating fine scale scaling functions to coarse scale scaling functions and wavelets):

$$\mathcal{S}_k = \mathcal{W}_{k-1} \oplus \mathcal{S}_{k-1}$$

$$s_n^{k-1}(x) = \underbrace{\sum_{l=0}^{2K-1} h_l s_{2n+l}^k(x)}_{\textbf{wavelet block-spin average}}$$

$$w_n^{k-1}(x) = \underbrace{\sum_{l=0}^{2K-1} g_l s_{2n+l}^k(x)}_{\textbf{lost short-distance information}}$$

$$g_l = (-)^l h_{2K-1-l}$$

Inverse relations

Reconstruct fine resolution scaling functions from coarse resolution scaling functions and wavelets

$$s_n^k(x) = \sum_m h_{n-2m} s_m^{k-1}(x) + \sum_m g_{n-2m} w_m^{k-1}(x)$$

Wavelet localized fields (scale k - one dimension for illustration)

$$\mathbf{\Phi}(x,t), \qquad \mathbf{\Pi}(x,t)$$

$$\mathbf{\Phi}_s^k(n,t) := \int s_n^k(x)\mathbf{\Phi}(x,t)dx$$

$$\mathbf{\Phi}_w^l(n,t) := \int w_n^l(x)\mathbf{\Phi}(x,t)dx$$

$$\mathbf{\Pi}_s^k(n,t) := \int s_n^k(x)\mathbf{\Pi}(x,t)dx$$

$$\mathbf{\Pi}_w^l(n,t) := \int \mathrm{w}_n^l(x)\mathbf{\Pi}(x,t)dx$$

Commutation relations (fixed scale k)

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y})$$

$$\Downarrow$$

$$[\mathbf{\Phi}_s^k(m,t), \mathbf{\Phi}_s^k(n,t)] = 0 \qquad [\mathbf{\Pi}_s^k(m,t), \mathbf{\Pi}_s^k(n,t)] = 0$$

$$[\mathbf{\Phi}_s^k(m,t), \mathbf{\Pi}_s^k(n,t)] = i\delta_{mn}$$

$$[\mathbf{\Phi}_w^k(m,t), \mathbf{\Phi}_w^l(n,t)] = 0 \qquad [\mathbf{\Pi}_w^k(m,t), \mathbf{\Pi}_w^l(n,t)] = 0$$

$$[\mathbf{\Phi}_w^k(m,t), \mathbf{\Pi}_w^l(n,t)] = i\delta_{mn}\delta_{kl}$$

$$[\mathbf{\Phi}_w^k(m,t), \mathbf{\Phi}_s^k(n,t)] = 0 \qquad [\mathbf{\Pi}_w^k(m,t), \mathbf{\Pi}_s^k(n,t)] = 0$$

$$[\mathbf{\Phi}_w^k(m,t), \mathbf{\Pi}_s^k(n,t)] = 0$$

- **Two-kinds of fields: scaling function fields, $\mathbf{\Phi}_s^k(n,t)$, and wavelet fields, $\mathbf{\Phi}_w^l(n,t)$.**

- **Scaling function fields encode scale $\frac{1}{2^k}$ physics.**

- **Wavelet fields encode fine resolution physics ($\frac{1}{2^l}$ $l \geq k$).**

- **Limiting l gives a short-distance cutoff.**

- **Limiting $-N \leq n \leq N$ gives a volume cutoff.**

- **Wavelet $+$ scaling function fields form a local algebra.**

Exact multi-resolution representation of field

$$\mathbf{\Phi}(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Phi}_s^k(n,t) s_n^k(x) + \sum_{n=-\infty}^{\infty} \sum_{l=k}^{\infty} \mathbf{\Phi}_w^l(n,t) w_n^l(x)$$

$$\mathbf{\Pi}(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Pi}_s^k(n,t) s_n^k(x) + \sum_{n=-\infty}^{\infty} \sum_{l=k}^{\infty} \mathbf{\Pi}_w^l(n,t) w_n^l(x)$$

Resolution $1/2^m$ $(m > k)$ field operators

$$\mathbf{\Phi}^m(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Phi}_s^m(n,t) s_n^m(x) =$$

$$\sum_{n=-\infty}^{\infty} \mathbf{\Phi}_s^k(n,t) s_n^k(x) + \sum_{n=-\infty}^{\infty} \sum_{l=k}^{m-1} \mathbf{\Phi}_w^l(n,t) w_n^l(x)$$

$$\mathbf{\Pi}^m(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Pi}_s^m(n,t) s_n^k(x) =$$

$$\sum_{n=-\infty}^{\infty} \mathbf{\Pi}_s^k(n,t) s_n^k(x) + \sum_{n=-\infty}^{\infty} \sum_{l=k}^{m-1} \mathbf{\Pi}_w^l(n,t) w_n^l(x)$$

Creation and annihilation operators are associated with each type of field

$$\mathbf{a}_s^k(n,t) := \frac{1}{\sqrt{2}}(\sqrt{\gamma}\mathbf{\Phi}_s^k(n,t) + i\frac{1}{\sqrt{\gamma}}\mathbf{\Pi}_s^k(n,t))$$

$$\mathbf{a}_w^k(n,t) := \frac{1}{\sqrt{2}}((\sqrt{\gamma}\mathbf{\Phi}_w^k(n,t) + i\frac{1}{\sqrt{\gamma}}\mathbf{\Pi}_w^k(n,t))$$

The coefficients γ are fixed by the requirement that the annihilation operators annihilate the vacuum:

$$\langle 0|\mathbf{a}_s^{k\dagger}(n,t)\mathbf{a}_s^{k}(n,t)|0\rangle = 0$$

$$\langle 0|\mathbf{a}_w^{k\dagger}(n,t)\mathbf{a}_w^{k}(n,t)|0\rangle = 0$$

$$\langle 0|\mathbf{a}_s^{k}(n,t)\mathbf{a}_s^{k\dagger}(n,t)|0\rangle = 1$$

$$\langle 0|\mathbf{a}_w^{k}(n,t)\mathbf{a}_w^{k\dagger}(n,t)|0\rangle = 1$$

Resolution k operator products

$$\int dx \mathbf{\Phi}^k(x)\mathbf{\Phi}^k(x)\mathbf{\Phi}^k(x)\mathbf{\Phi}^k(x) =$$

$$\sum \mathbf{\Phi}_s^k(n_1,t)\mathbf{\Phi}_s^k(n_2,t)\mathbf{\Phi}_s^k(n_3,t)\mathbf{\Phi}_s^k(n_4,t)\Gamma^k_{n_1 n_2 n_3 n_4}$$

where

$$\Gamma^k_{n_1 n_2 n_3 n_4} := \int s^k_{n_1}(x)s^k_{n_2}(x)s^k_{n_3}(x)s^k_{n_4}(x)dx$$

and

$$\int dx \frac{\partial}{\partial x}\mathbf{\Phi}^k(x)\frac{\partial}{\partial x}\mathbf{\Phi}^k(x) = \sum \mathbf{\Phi}_s^k(n_1,t)\mathbf{\Phi}_s^k(n_2,t)D^k_{n_1 n_2}$$

where

$$D^k_{mn} = \int dx \frac{\partial}{\partial x}s^k_m(x)\frac{\partial}{\partial x}s^k_n(x)$$

Numerical coefficients have simple scaling properties

$$\Gamma^k_{n_1 \cdots n_m} = 2^{k(m-2)/2} \Gamma^0_{n_1 \cdots n_m}$$

$$D^k_{mn} = 2^k D^0_{mn}$$

$\Gamma^0_{n_1 \cdots n_m}$ and D^0_{mn} generated using translational invariance from solutions of a finite linear systems with known coefficients.

RG computation of Γ and D

$$\Gamma^0_{0 n_2 n_3} = \sqrt{2} \sum h_{l_1} h_{l_2} h_{l_3} \Gamma^0_{0 2n_2+l_2-l_1, 2n_3+l_3-l_1}$$

$$\sum \Gamma^0_{0 n_2 n_3} = \delta_{n_2 0}$$

$$D^0_{0,n_1} = \sum 4 h_{l_1} h_{l_2} D^0_{0,l_2 n_2+l_2-l_1}$$

Hamiltonian

$$H =$$

$$\frac{1}{2}\int :(\mathbf{\Pi}(x)^2 + \nabla\mathbf{\Phi}(x)\cdot\nabla\mathbf{\Phi}(x) + \mu^2\mathbf{\Phi}(x)^2 + \lambda\mathbf{\Phi}(x)^4): dx$$

Resolution $1/2^k$ Hamiltonian

$$\mathbf{\Phi}(x) \to \mathbf{\Phi}_s^k(x) \qquad \mathbf{\Pi}(x) \to \mathbf{\Pi}_s^k(x)$$

$$H^k =$$

$$\frac{1}{2}\sum : \Big(\mathbf{\Pi}_s^k(n,0)^2 + D_{mn}^k \mathbf{\Phi}_s^k(n,0)\mathbf{\Phi}_s^k(m,0) + \mu^2\mathbf{\Phi}_s^k(n,0)^2 +$$

$$\lambda\Gamma^k_{n_1 n_2 n_3 n_4}\mathbf{\Phi}_s^k(n_1,0)\mathbf{\Phi}_s^k(n_2,0)\mathbf{\Phi}_s^k(n_3,0)\mathbf{\Phi}_s^k(n_4,0)\Big) :$$

Two scale Hamiltonian

$$\mathbf{\Phi}_s^{k+1}(n,0) = \sum_m (h_{n-2m}\mathbf{\Phi}_s^k(n,0) + g_{n-2m}\mathbf{\Phi}_w^k(n,0))$$

$$\Gamma^k_{n_1 n_2 n_3 n_4} \to \Gamma^{k+1}_{n_1 n_2 n_3 n_4} = 2\Gamma^k_{n_1 n_2 n_3 n_4}$$

$$D^k_{mn} \to D^{k+1}_{mn} = 2D^k_{mn}$$

This leads to separation of scales $1/2^{k+1}$ and $1/2^k$ and a coupling term

$$H_s^{k+1}(\mathbf{a}_s^{k+1}, \mathbf{a}_s^{k+1\dagger}) =$$

$$H_s^k(\mathbf{a}_s^k, \mathbf{a}_s^{k\dagger}) + H_w^k(\mathbf{a}_w^k, \mathbf{a}_w^{k\,\dagger}) + H_{sw}^k(\mathbf{a}_s^k, \mathbf{a}_s^{k\dagger}, \mathbf{a}_w^k, \mathbf{a}_w^{k\,\dagger})$$

RG

- H^k and H^{k+1} have the same form with rescaled coefficients.
- Eliminating the wavelet degrees of freedom gives a new $H^k(1)$ involving the same parameters and same scale $1/2^k$ degrees of freedom including effects eliminated scale $1/2^{k+1}$ degrees of freedom.
- Readjust parameters to fix scale $1/2^k$ observables.
- Rescaling the coefficients $\Gamma^k \to \Gamma^{k+1}$, $D^k \to D^{k+1}$ gives scale $H^{k+1}(1)$ Hamiltonian including effects of eliminated scale $k+2$ degrees of freedom.
- This process can be repeated to determine evolution of bare coupling constants as a function of resolution k.

Elimination can be performed using the similarity renormalization group method.

$$\frac{dH(\lambda)}{d\lambda} = [H(\lambda), [H(\lambda), H_s^k + H_w^k]]$$

Approximations can be made because the size of all coefficients are known

Partitions of unity and symmetries

$$[O^a(x), O^b(y)] = i\delta(x-y) f^{abc} O^c(y)$$

$$1 = (2^{-k/2} \sum_n s_n^k(x))(2^{-k/2} \sum_m s_m^k(y))$$

$$1 = (2^{-k/2} \sum_n s_n^k(x))$$

$$O_n^{ak} := 2^{k/2} \int O^a(x) s_n^k(x) dx$$

$$[\sum_n O_n^{ak}, \sum_m O_m^{bk}] = i f^{abc} \sum_l O_l^{ck}$$

$$O^a \to \sum_n O_n^{ak}$$

Gives local generators - the symmetry is broken when products of finite resolution discrete fields are used to construct O_n^{ak}.

Truncations as approximations

$$\|[\mathbf{\Phi}(f_1)\cdots\mathbf{\Phi}(f_n)|0\rangle - \mathbf{\Phi}^k(f_1)\cdots\mathbf{\Phi}^k(f_n)|0\rangle]\| = \Delta(k)$$

$$\mathbf{\Phi}(f) \to \int :\mathbf{\Phi}^n(x): f(x)dx$$

Summary

- Wavelet methods provided a useful representation for understanding problems in quantum field theory.

- Formulation of RG equations

- Test of symmetries

- Tests of truncations as approximations