Systematic regularization (and renormalization) scheme in non-perturbative calculations within covariant light-front dynamics

Jean-François Mathiot

Laboratoire de Physique Corpusculaire
Clermont-Ferrand, France

UNIVERSITE BLAISE PASCAL CLERMONT-FERRAND

in collaboration with

V. Karmanov, A. Smirnov
P. Grangé, E. Werner

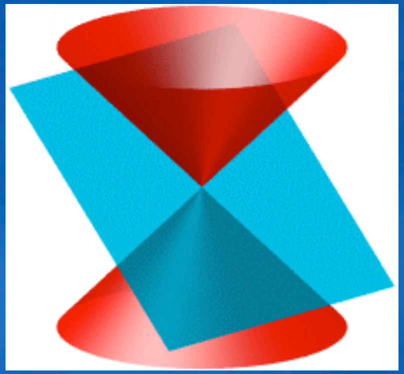
cnrs dépasser les frontières

PLAN

- Covariant light-front dynamics
- Non-perturbative calculations
- Renormalization conditions
- Systematic regularization scheme
- Conclusions

Covariant light-front dynamics

❑ Control of rotational invariance



➢ arbitrary position ω of the light-front

$$\tau = \omega.x \quad \text{with} \quad \omega^2 = 0$$

➢ control of rotational invariance at each order of perturbation theory or at each order of any approximative scheme (ex. Fock space truncation)

➢ Either calculated physical observables are rotational invariant, or we can extract the physical, ω -independent contribution

ex. renormalization condition

V. Karmanov 1976, J. Carbonell et al. Phys. Rep. 300 (1998) 215

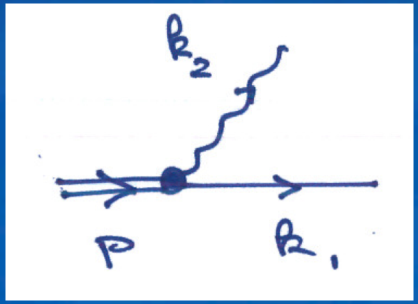
❑ Sources of violation of rotational invariance

Two very different origins in an ab-initio calculation

- ➢ **Regularization scheme**
  - ➥ **ex. cut-off in transverse momentum space**
  - ➥ **use of rotational invariant regularization scheme**

- ➢ **Truncation of the Fock space (or any other approximative scheme which does not preserve rotational invariance)**
  - ➥ **use of appropriate counterterms to restore rotational invariance if needed**

❑ Off-shell energy/momentum



$$p^2 = M^2 \qquad k_1^2 = m^2 \qquad k_2^2 = \mu^2$$

$$p = k_1 + k_2 - \omega\tau$$

$$\tau = \frac{s - M^2}{2\ \omega.p} \quad \textbf{with} \quad s = (k_1 + k_2)^2$$

➢ τ **fixed** **from the kinematics of the** **two-body state**

➢ **systematic calculations** **of energy/momentum conservation**

Non-perturbative calculations

- Poincaré group equations

$$P^2 \phi(p) = M^2 \phi(p)$$

$\phi(p)$: state vector

- Fock space expansion

$$\Phi = \sum_i \phi_i(k_1 \ldots k_i)$$

➢ Truncation of the Fock space (to order N) for obvious technical reasons

➢ System of coupled equations which sums to all orders a given class of irreducible contributions with up to N particles in the intermediate states

- Non-perturbative quantity ϕ_i or more conveniently Γ_i

  - $\phi_i = \dfrac{\Gamma_i}{s_i - M^2}$ $\qquad \Gamma_i$: vertex function

    $s_i = (k_1 + \ldots k_i)^2$

  - ex. $\Gamma_2(x, k_\perp^2)$ with $\Gamma_2(x, k_\perp^2) \to cte$ for $k_\perp^2 \to \infty$

- Covariant decomposition of Γ_i

  - $\Gamma_2 = a_2 + b_2 \dfrac{M \not{\omega}}{\omega\cdot p}$ for the Yukawa model

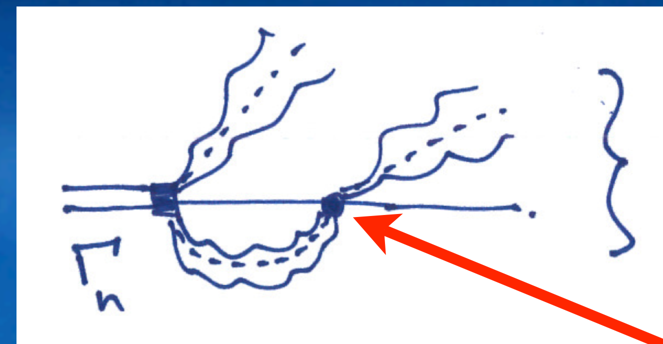
    $$\Gamma_3 = a_2 + b_2 \frac{M \not{\omega}}{\omega\cdot p} + C_{ps}\left(c_2 + d_2 \frac{M \not{\omega}}{\omega\cdot p}\right)\gamma_5$$

    $$C_{ps} = i\frac{1}{M^2 \omega\cdot p}\epsilon_{\mu\nu\rho\sigma}\ k_2^\mu\ k_3^\nu\ p^\sigma\ \omega^\rho$$

  - in general $\Gamma_n = \sum_i a_i\ O_i$ ← vertex operators

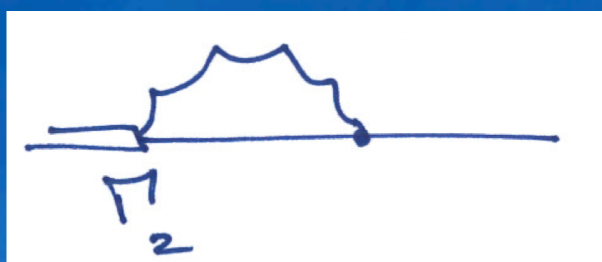
❑ General contribution

➢ general covariant decomposition



off-shell momentum k

H_{int}



➢ simplest non-perturbative contribution in the Yukawa model

$$A(k^2) + B(k^2)\,\frac{\not k}{M} + C(k^2)\frac{M\,\not\omega}{\omega\cdot k} + D(k^2)\frac{\not\omega\,\not k - \not k\,\not\omega}{\omega\cdot k}$$

$$A(k^2) \sim \int dx\; dk_\perp^2 \frac{a_2(x,k_\perp^2)}{k_\perp^2 + M^2 x + \mu^2(1-x) - k^2 x(1-x)}$$

up to a regularization scheme

Renormalization scheme

- **Physical mass**

physical mass

$$P^2\phi(p) = M^2\phi(p)$$

➢ Solution of the homogeneous coupled equations defines the appropriate (non-perturbative) mass counterterm

- **Physical coupling constant**

➢ using the standard procedure

physical coupling constant

$$\Gamma_2(s_2 = M^2) = g\sqrt{Z_f}$$

(in the quenched approximation)

➢ rem. $\Gamma_2(s_2 = M^2)$ should be independent of x (cf V. Karmanov)

➢ $\Gamma_2(s_2 = M^2)$ should be also independent of the position of the light front

➢ necessary condition to define the PHYSICAL coupling constant g

- one **should extract** (or remove with an appropriate counterterm) the ω - independent contribution to $\Gamma_2(s_2 = M^2)$

- this is **only possible** in the **covariant formulation** of light-front dynamics (unless the calculation is exact!)

$$\Gamma_2(s_2 - M^2) = a_2(s_2 - M^2) + b_2(s_2 - M^2)\ \frac{M\ \not{\omega}}{\omega\cdot p}$$

$a_2 \leftarrow g\sqrt{Z_f}$

- defines the **minimal number** of counterterms to be introduced

Systematic regularization scheme

- The key role of non-perturbative regularization scheme
  - first approach with Pauli-Villars boson and fermion fields
    - no contact interactions
    - infinite mass limit for both PV fermion and boson
    - delicate numerical cancellations
  - need for a more systematic approach which preserves rotational invariance and which does not involve infinitely large energy/momentum scales
  - Taylor-Lagrange regularization scheme

❑ Reminder of the Taylor-Lagrange regularization scheme

P. Grangé, E. Werner (2006)

➢ Fields defined as **distributions** acting on (super-regular) test functions of **space-time extension a**

$$\varphi_a(x) = \int \frac{d^3p}{(2\pi)^3} \frac{f_a(\epsilon_p^2, \mathbf{p}^2)}{2\epsilon_p} \left[a_p^\dagger e^{i\mathbf{p.x}} + a_p e^{-i\mathbf{p.x}}\right]$$

➢ Use of the **Lagrange formula** (in one dimension X for a simple logarithmic divergence)

$$f_a(\lambda X) = -\int_\lambda^\infty dt\ \partial_t\ f_a(Xt)$$

λ : intrinsic scale

➢ Continuum limit $a \rightarrow 0$ or in momentum space cut-off $H \rightarrow \infty$

ultra-soft cut-off

$$H \rightarrow H(X) \equiv \eta^2\ X^\alpha\ +\ cte \qquad \eta^2 > 1 \qquad 0 \leq \alpha < 1$$

$H \rightarrow \infty$ for $\alpha \rightarrow 1$

- ➢ Arbitrary scale η plays a very similar rôle to the unit of mass μ in dimensional regularization: completely arbitrary and physical observables should not depend on it

- ➢ for an amplitude of the form $\mathcal{A}_a = \int dX \; T(X) \; f_a(X)$

  and a singular operator $T(X) = \frac{1}{X + \lambda}$

- ➢ integration by part and change of variable $Y = \frac{X}{\lambda} t$

$$\mathcal{A} = \int dY \int_\lambda^{\eta^2} \partial_t \left[\frac{1}{t} \, T\left(\frac{Y}{t} \right) \right] \; f(Y)$$

in the continuum limit $a \longrightarrow 0$

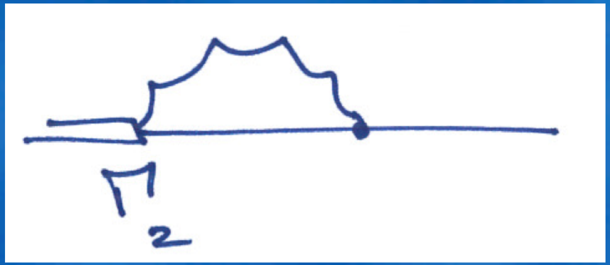
$$\mathcal{A} = \int dY \;\; f(Y) \left[\frac{1}{Y + \eta^2} - \frac{1}{Y + \lambda} \right]$$

❑ Application to non-perturbative calculations in light-front dynamics

➢ **very simple implementation** of the Taylor-Lagrange regularization scheme in covariant light-front dynamics

$$\Gamma_n(k_1 \ldots k_n) \rightarrow \bar{\Gamma}_n(k_1 \ldots k_n) = \Gamma_n(k_1 \ldots k_n) f(k_1^2) \ldots (k_n^2)$$

➢ the **properties of the test functions** are now embedded in $\bar{\Gamma}_n$ and in its covariant components



➢ **simplest non-perturbative contribution**
in the Yukawa model

$\bar{a}_2$

$$A(k^2) \sim \int dx\ dk_\perp^2 \frac{a_2(x, k_\perp^2) f_a\left[\left(\frac{k_\perp^2}{2xM\Lambda}\right)^2\right]}{k_\perp^2 + M^2 x + \mu^2(1-x) - k^2 x(1-x)}$$

Λ : arbitrary momentum scale

➢ with the identification $X = \frac{k_\perp^2}{2xM\Lambda}$

➢ and the intrinsic scale $\lambda(x) = \frac{M^2 x + \mu^2(1-x) - k^2 x(1-x)}{M^2}$

➢ one gets

$$A(k^2) \sim \int dx\ dk_\perp^2 \left[\frac{1}{k_\perp^2 + M^2\ \eta^2\ \lambda(x)} - \frac{1}{k_\perp^2 + M^2\ \lambda(x)^2} \right] a_2(x, k_\perp^2)$$

➢ with $a_2 = 1$ one recovers the perturbative result in terms of the arbitrary scale η^2

$$A(k^2) \sim \int dx\ \left(Log[\eta^2] - Log[\lambda(x)] \right)$$

Conclusions

- The general framework is now settled

- First non-perturbative calculations in the Yukawa model
  - show that it is indeed applicable and successfull in the Yukawa model
  - extension to N > 3 to be done (see also V. Karmanov)

- Main difficulty : application to gauge theories
  - expansion of the vertex functions in covariant components may become rather tedious
  - is it absolutely necessary? (see also V. Karmanov and J. Vary)

Construction of the physical fields

❑ Definition of the physical fields

ex.: scalar field $\phi(x)$

- ➢ Fields should be considered as distributions
- ➢ Functional Φ with respect to a test function ρ

$$\Phi(\rho) = \int d^4y \; \phi(y) \; \rho(y)$$

N. Bogoliubov, 1950's

- ➢ Physical field $\varphi(x)$ by means of the translation operator T_x

$$\varphi(x) \equiv T_x \Phi(\rho) = \int d^4y \; \phi(y) \; \rho(x-y)$$

❑ Properties of the test functions

- ➢ belongs to the Schwartz space $\mathcal{S}$ of fast decrease functions
  - ➥ decrease at infinity faster than any power of x, as well as all its derivatives
  - ➥ property conserved by Fourier transform

- in momentum space

$$\rho(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)} f(p_0^2, \mathbf{p}^2)$$

- decomposition of the physical field

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{f(\epsilon_p^2, \mathbf{p}^2)}{2\epsilon_p} \left[a_p^\dagger e^{i\mathbf{p}.\mathbf{x}} + a_p e^{-i\mathbf{p}.\mathbf{x}}\right]$$

❑ Physical interpretation of the test function

- $\varphi(x)$: average over the initial field with a weight ρ
  - ➥ if ρ has a space-time extension a : average over a volume a^4

$$\rho_a(x) \longrightarrow \varphi_a(x)$$

- to recover a "local" field theory, one should investigate the limit $a \longrightarrow 0$
- scale invariance inherent to this limit since also $\frac{a}{\eta} \longrightarrow 0$ with $\eta > 1$

so that a priori $\rho_a(x) \longrightarrow \rho_\eta(x)$ and $\varphi_a(x) \longrightarrow \varphi_\eta(x)$

- for the Fourier transform of ρ_a

$$f_a \overset{a \to 0}{\longrightarrow} f_\eta \sim cte$$

- it is sufficient to consider $f_\eta \sim 1$

➥ Poincaré group equations invariant without renormalization of the fields

- calculation of any amplitude

$$\mathcal{A}_\eta = \int dX \; T(X) \; f_\eta(X)$$

with a one dimensional variable X for simplicity

ex.: $X = \dfrac{k_E^2}{\Lambda^2}$, Λ arbitrary scale

$T(X)$: singular distribution : $\mathcal{A}_\eta$ divergent if no test functions

Explicit construction of the test function

- we shall first consider a sequence of test functions f_α with compact support

$$f_\alpha(H) = 0 \quad \text{, with} \quad H \equiv X_{max}$$

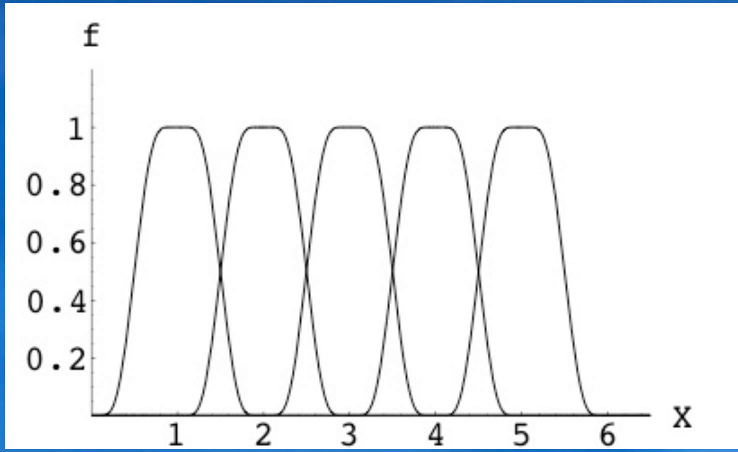
so that

$$\mathcal{A}_\alpha = \int dX \; T(X) \; f_\alpha(X)$$

- f_α chosen as a partition of unity (PU)
  - ➥ $\mathcal{A}_a$ independent of the particular choice of a PU

- construction of a PU

$$f(x) = \sum_{j=0}^{N-1} u(x - jh)$$



- in a given limit $\alpha \to 1^-$ $\quad f_\alpha(x) \to 1$

- in this limit, one should recover the original test function

$$\lim_{\alpha \to 1^-} \mathcal{A}_\alpha \equiv \mathcal{A}_\eta$$

➥ This limit should be independent of X_{max}

➢ one needs a **particular construction** of the test function f_α

➥ **Ultra-soft cut-off** (“dynamical” cut-off)

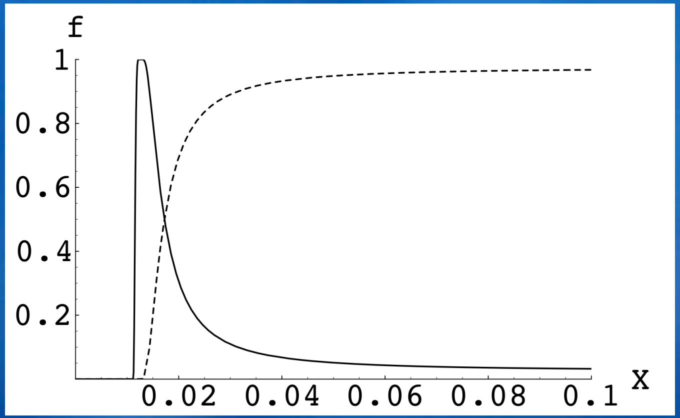
$$H \rightarrow H(X) \equiv \eta^2 \, X^\alpha \; + \; cte \qquad\qquad \eta^2 > 1$$

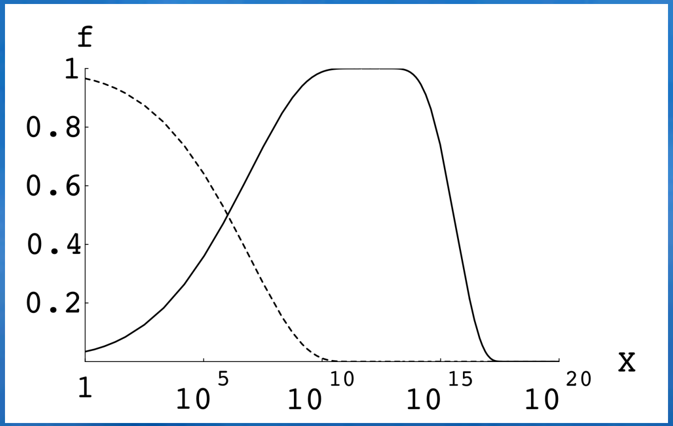
Rem.: not at all unique example

➥ **upper limit** of f_α defined by $X_{max} = H(X_{max})$

$$X_{max} = (\eta^2)^{\frac{1}{1-\alpha}}$$

$$\lim_{\alpha \rightarrow 1^-} X_{max} = \infty$$





➢ **the Taylor-Lagrange regularization scheme**

Construction of (finite) extended bare amplitudes

❑ Extension in the ultra-violet domain

➢ Apply the **Lagrange formula** for the Taylor remainder of $f_\alpha = R_k\ f_\alpha$

$$f(\lambda X) = -\frac{X}{\lambda^k k!}\int_\lambda^\infty \frac{dt}{t}(\lambda - t)^k \partial_X^{k+1}\left[X^k f(Xt)\right]$$

λ **intrinsic scale** ex.: $T(X) = \dfrac{1}{X+\lambda}$

➢ one should thus calculate $\mathcal{A}_\alpha = \int_0^\infty dX\ T(X)\ f_\alpha(X)$ $\quad \alpha \to 1^-$

➢ by **integration by part** after use of the Lagrange formula

$$\mathcal{A}_\alpha = \int_0^\infty dX\ \tilde{T}_\alpha^{>}(X)\ f_\alpha(X)$$

In the limit $\alpha \to 1^-$, $\tilde{T}^{>}_{\alpha}(X) \to \tilde{T}^{>}_{\eta}(X)$ with

$$T^{>}_{\eta}(X) = \frac{(-X)^k}{\lambda^k k!} \partial_X^{k+1} \left[X T(X) \right] \int_{\lambda}^{\eta^2} \frac{dt}{t} (\lambda - t)^k$$

➢ because of the **derivatives** in $\tilde{T}_{\eta}(X)$, the amplitude is now **completely finite**

$$\mathcal{A}_{\alpha} \to \mathcal{A}_{\eta} = \int_0^{\infty} dX\ \tilde{T}^{>}_{\eta}(X)$$

➥ depends on the **arbitrary scale** η^2

➥ if $T(X) = \dfrac{1}{X + \lambda}$ $\quad \tilde{T}^{>}_{\eta}(X) = \mathrm{Ln}\left(\dfrac{\eta^2}{\lambda}\right)$

❑ Extension in the infra-red domain

➢ Typical distribution $T^{<}(X) = \dfrac{1}{X^{k+1}}$ with no intrinsic scale

➢ **extended** distribution

$$\tilde{T}^{<}(X) = \frac{(-1)^k}{k!} \partial_X^{k+1}\ \mathrm{Ln}(\tilde{\eta} X) \equiv Pf\left[\frac{1}{X^{k+1}}\right]$$