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Present status of the Bethe-Salpeter approach in Minkowski space

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Bethe-Salpeter approach provides a covariant description of relativistic few-body systems from the first principles.

• Plan

- Why Minkowski space?
- Bound states. Solving BS equation via Nakanishi representation.
  - Spinless particles.
  - Fermions.
  - Electromagnetic form factor.
- Scattering states (by another method).
- What should be done?

• Definition

E.E. Salpeter, H.A. Bethe, Phys. Rev. **84** (1951) 1232.

$$\Phi(x_1, x_2, p) = \langle 0 | T \left\{ \varphi(x_1) \varphi(x_2) \right\} | p \rangle, \quad p^2 = M^2$$

$$\Phi(x_1, x_2, p) = \exp(-i(x_1 + x_2)p/2)\Phi(x_1 - x_2, p), \quad x = x_1 - x_2$$

$$\Phi(k, p) = \int \exp(ikx)\Phi(x, p) d^4x$$

• Variables

$$\Phi(k,p) \Rightarrow \Phi(k_1,k_2,p), \quad k_1+k_2=p,$$

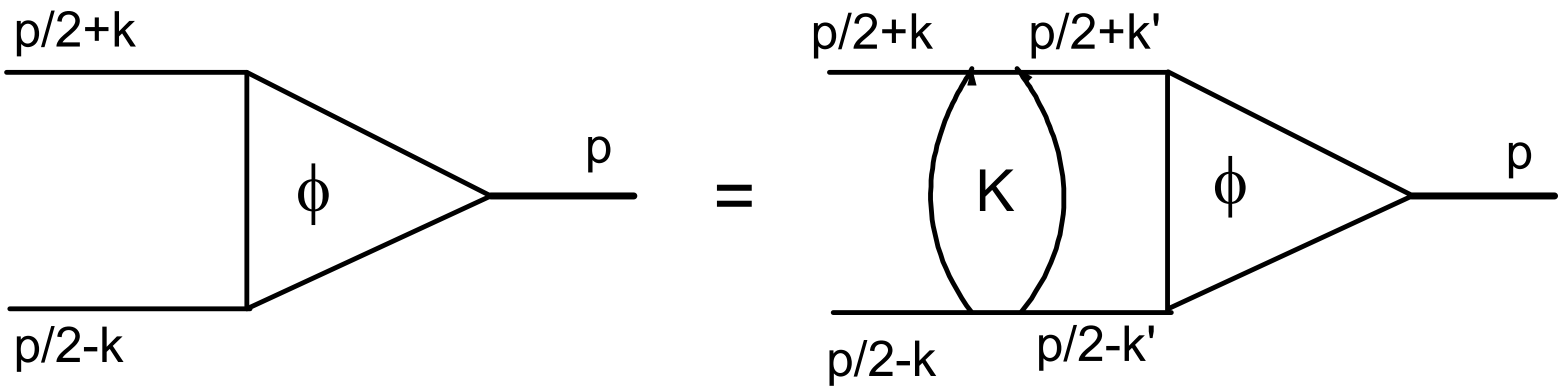
$$k_1^2 \neq m_1^2, \quad k_2^2 \neq m_2^2, \quad p^2 = M^2$$

$\Rightarrow$ off-mass shall amplitude

Or:

$\Phi(k,p) \Rightarrow \Phi(k^2, k\cdot p) \rightarrow$ in c.m.-frame $\vec{p}=0: \quad \Phi = \Phi(|\vec{k}|, k_0)$

• BS bound state equation



$$\Phi(k,p) = \frac{-i}{\left((\frac{p}{2}+k)^2 - m^2 + i\epsilon\right)\left((\frac{p}{2}-k)^2 - m^2 + i\epsilon\right)}$$

$$\times \int \frac{d^4k'}{(2\pi)^4} K(k,k',p)\Phi(k',p)$$

BS equation contains singular integrals!

It is not a problem in principle (it is normal).

But it is a problem for numerical solution.

• Wick rotation ⇒ Euclidean space

Propagators:

$$\frac{1}{\left(k_1^2 - m^2 + i\epsilon\right)} \; \frac{1}{\left(k_2^2 - m^2 + i\epsilon\right)} =$$

$$\frac{1}{\left((\frac{p}{2} + k)^2 - m^2 + i\epsilon\right)} \; \frac{1}{\left((\frac{p}{2} - k)^2 - m^2 + i\epsilon\right)}$$

$\Rightarrow \quad k_0 \to ik_4,$ c.m.-frame $\Rightarrow$

$$\frac{1}{\left(m^2 - \frac{M^2}{4} + \vec{k}^2 + k_4^2\right)^2 + M^2 k_4^2}$$

Kernel (OBE):

$$K = \frac{-g^2}{(k - k')^2 - \mu^2} \Rightarrow K_E = \frac{g^2}{(\vec{k} - \vec{k'})^2 + (k_4 - k_4')^2 + \mu^2}$$

BS amplitude:

$$\Phi(\vec{k},k_0) \to \Phi_E(\vec{k},k_4) = \Phi(\vec{k},ik_0)$$

Equation:

$$\Phi_E(\vec{k},k_4) = \frac{1}{\left(m^2-\frac{M^2}{4}+\vec{k}^2+k_4^2\right)^2+M^2k_4^2} \times \int \frac{d^3k'dk_4}{(2\pi)^4} K_E(k,k')\Phi_E(\vec{k}',k_4)$$

It gives exactly the same binding energies as in Minkowski space.

Finding solution in Euclidean space is a trivial numerical task. We find $\Phi_E(\vec{k},k_4)$ and M.

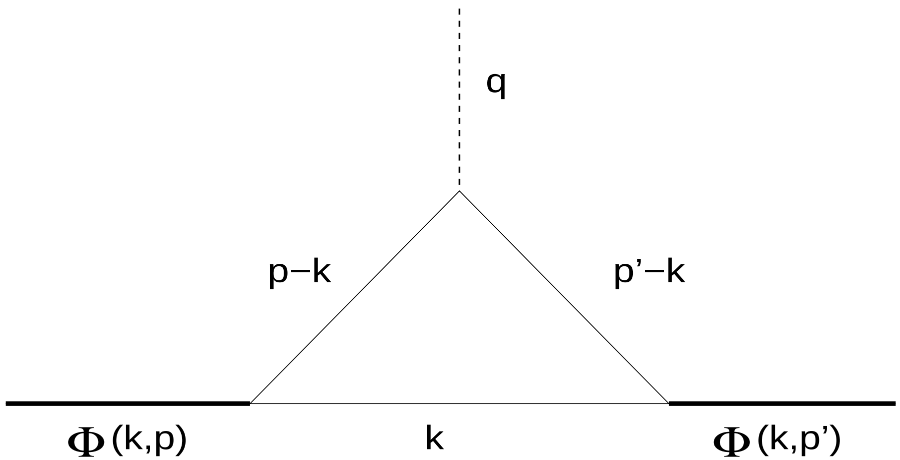
But we cannot extrapolate $\Phi_E(\vec{k},k_4)$ via complex plane back in Minkowski space numerically.

Unstable extrapolation: $\Phi_E(\vec{k},k_4) \to \Phi_E(\vec{k},-ik_4) = \Phi(\vec{k},k_0)$

Therefore we cannot use $\Phi_E(\vec{k},k_4)$.

Or we should find the BS amplitude in the complex plane, P. Maris, P. Tandy, (2006).

• Why Minkowsky space?



E.m. vertex in terms of the BS amplitude.

$$(p+p')^\mu F_M(Q^2) = -i\int \frac{d^3k\,dk_0}{(2\pi)^4}(p+p'-2k)^\mu\;(m^2-k^2)\Phi_M\left(\frac{p}{2}-k,p\right)\Phi_M\left(\frac{p'}{2}-k,p'\right)$$

Wick rotation cannot be done because (of infinite number) of singularities of Φ_M vs. k_0.

• Example

BS amplitude: $$\Phi_M = \frac{1}{[k^2 - m^2 + i\epsilon][(p-k)^2 - m^2 + i\epsilon]}$$

E.M. form factor:

$$4M^2 F_M(0) = i \int \frac{d^4k}{(2\pi)^4} \frac{(4M^2 - 4k\cdot p)}{[k^2 - m^2 + i\epsilon][(p-k)^2 - m^2 + i\epsilon]^2}$$

Feynman parametrization: $$\frac{1}{ab^2} = \int_0^1 \frac{2x dx}{[(1-x)a + xb]^3}$$

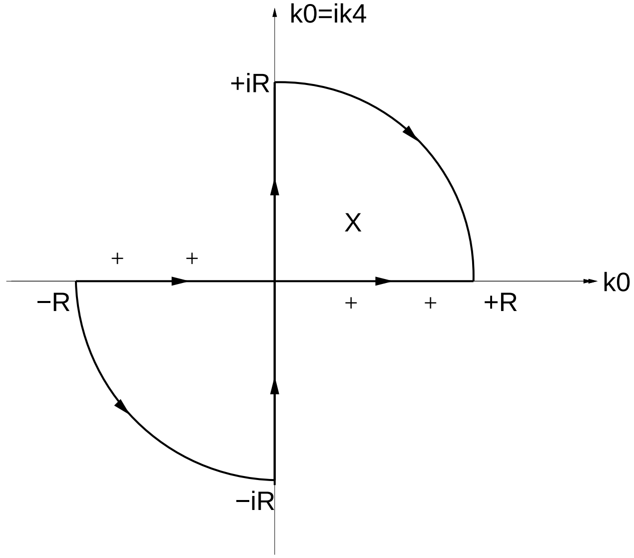
Result:

$$F_M(0) = \frac{1}{16\pi^2 M^3}\left(\frac{4m^2 \arctan \frac{M}{\sqrt{4m^2-M^2}}}{\sqrt{4m^2 - M^2}} - M\right)$$

For $m = 2$ and $M = 3$: $F_M(0) = 4.99241 \times 10^{-4}$.

• Naive Wick rotation $k_0 \to ik_4$

(Neglecting singularity)



Wick rotation in the form factor integral.

$$F_{NW}(0) = \frac{-4\pi i}{(2\pi)^4 M} \int_0^\infty k^2 dk \int_{-\infty}^{\infty} \frac{(M - ik_4) i dk_4}{[k_4^2 + \vec{k}^2 + m^2][(k_4 + iM)^2 + \vec{k}^2 + m^2]^2}$$

$$F_{NW}(0) = 3.15404 \times 10^{-4} \neq F_M(0) = 4.99241 \times 10^{-4}$$

• Contribution of singularity

Position of singularity:

$$k_0 = M - \sqrt{\vec{k}^2 + m^2} + i\epsilon$$

Residue:

$$\mathrm{Res} = \frac{-i(M - \sqrt{\vec{k}^2 + m^2})}{2^5\pi^2 M^3 \sqrt{\vec{k}^2 + m^2}(2\sqrt{\vec{k}^2 + m^2} - M)^2} \qquad \text{(if } k^2 < M^2 - m^2\text{)}$$

Residue contribution:

$$F_{Res}(0) = \int_0^{\sqrt{M^2 - m^2}} (2\pi i\,\mathrm{Res})\, 4\pi k^2 dk = 1.83837 \times 10^{-4}$$

Full result:

$$F_{NW}(0) + F_{Res}(0) = 3.15404 \cdot 10^{-4} + 1.83837 \cdot 10^{-4} = 4.99241 \cdot 10^{-4} = F_M(0)$$

• Infinite number of singularities

$\Phi = \phi(k_1^2, k_2^2, p)$ is singular at $k_1^2 = (m+\mu)^2,\ (m+2\mu)^2, \ldots$
and at $k_2^2 = (m+\mu)^2,\ (m+2\mu)^2, \ldots$.

Since $k_1 = \frac{p}{2} + k$, $k_2 = \frac{p}{2} - k$, this gives infinite number of singularities of $\Phi(k, k_0)$ vs. k_0 which move when k varies.

They may be weak and approximatively omitted,
but we do not want to make uncontrolable approximations.

• Minkowski space solution

(first solution, for the ladder kernel only)

K. Kusaka, A.G. Williams, Phys.Rev. **D51** *(1995) 7026;*

K. Kusaka, K. Simpson, A.G. Williams, Phys.Rev. **D56** *(1997) 5071.*

To find solution, one should:

1) Represent BS amplitude through the Nakanishi integral

Nakanishi, N.: *Graph Theory and Feynman Integrals,*

Gordon and Breach, New York, 1971):

$$\Phi(k,p)=\int_{-1}^{1}dz'\int_0^{\infty}d\gamma'\frac{-ig(\gamma',z')}{\left[\gamma'+m^2-\frac{1}{4}M^2-k^2-p{\cdot}k\;z'-i\epsilon\right]^3}.$$

• Finding amplitude $\Phi(k,p)$

2) Substitute $\Phi(k,p)$ in the BS equation.
Derive and solve equation for $g(\gamma,z)$.

3) Substitute $g(\gamma,z)$ back in the Nakanishi integral and find BS amplitude $\Phi(k,p)$ in Minkowski space.

It works! However:

- Derivation of equation for $g(\gamma,z)$ depends on the kernel.
- Equation for $g(\gamma,z)$ is rather cumbersome.
- This equation was derived and solved for ladder kernel only.

• The case $g(\gamma', z') = 1$

$$\Phi(k,p) =$$

$$= \int_{-1}^{1} dz' \int_0^\infty d\gamma' \frac{-i'}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p{\cdot}k\ z' - i\epsilon\right]^3}$$

$$= \frac{-i}{\left((\frac{p}{2}+k)^2 - m^2 + i\epsilon\right)\ \left((\frac{p}{2}-k)^2 - m^2 + i\epsilon\right)}$$

All the non-trivial dynamics is in the non-singular function $g(\gamma', z')$.

• Another derivation

Karmanov, V.A., Carbonell, J., Eur. Phys. J. A **27**, 1 (2006)

- Take BS amplitude in the Nakanishi form:

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{-ig(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p{\cdot}k\; z' - i\epsilon\right]^3}$$

- Substitute it in the BS equation.
- First, apply to both sides of the BS equation the LF projection:

$$\psi(\vec{k}_\perp, x) = \int \Phi(k,p) dk_-$$

- Then obtain equation for $g(\gamma, z)$.

LF projection

$$
\begin{aligned}
\psi(\vec{k}_\perp, x) &= \int_{-\infty}^{\infty} \Phi(k,p)\, dk_- \\
&= \int_0^{\infty} \frac{g(\gamma', 1-2x)\, d\gamma'}{\left[\gamma' + \vec{k}_\perp^2 + m^2 - x(1-x)M^2\right]^2}
\end{aligned}
$$

Advantage:

Light-front wave function $\psi(\vec{k}_\perp, x)$ is non-singular.

• Equation for $g(\gamma, z)$

$$\int_0^\infty \frac{g(\gamma', z)d\gamma'}{\left[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2\right]^2}$$

$$= \int_0^\infty d\gamma' \int_{-1}^1 dz' \, V(\gamma, z; \gamma', z') g(\gamma', z')$$

where $z = 1 - 2x$, $\kappa^2 = m^2 - \frac{1}{4}M^2$.

- This equation is equivalent to the initial BS equation.

Matrix form:

$$\lambda B x = A x$$

It is just standard form for well known fortran subroutines.

• OBE kernel

One-boson exchange (ladder) kernel:

$$K(k,k',p)=\frac{-g^2}{(k-k')^2-\mu^2+i\epsilon}$$

where $g^2=16\pi m^2\alpha$.

After applying the Nakanishi integral and LF projection OBE kernel turns into the kernel $V(\gamma,z;\gamma',z')$:

$$V(\gamma,z;\gamma',z')=\frac{\alpha m^2(1-z)^2}{2\pi\Big[\gamma+z^2m^2+(1-z^2)\kappa^2\Big]}\int_0^1\frac{v^2dv}{B_1^2}$$

$B_1=B_1(\gamma,z;\gamma',z';v)$ is a polynomial.

Integral $\int_0^1\frac{v^2dv}{B_1^2}$ is calculated analytically.

• Kernel $V(\gamma, z; \gamma', z')$ v.s. z'

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(* Spinless case  *)
z = 0.8
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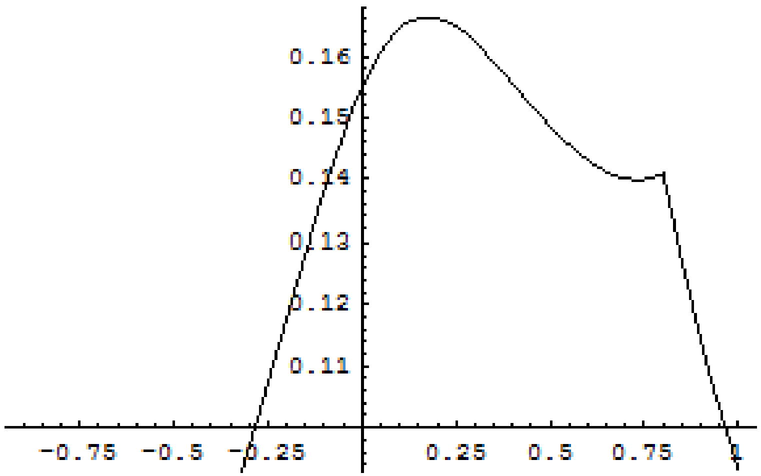


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- Graphics -
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Kernel $V(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.8$

• Equation

(reminder)

$$\int_0^\infty \frac{g(\gamma', z)d\gamma'}{\left[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2\right]^2}$$

$$= \int_0^\infty d\gamma' \int_{-1}^{1} dz' \, V(\gamma, z; \gamma', z') g(\gamma', z')$$

• Numerical results ($\mu \neq 0$)

Coupling constant $\alpha = \frac{g^2}{16\pi m^2}$ as a function of the binding energy for $\mu = 0.15$ and $\mu = 0.5$, $m = 1$

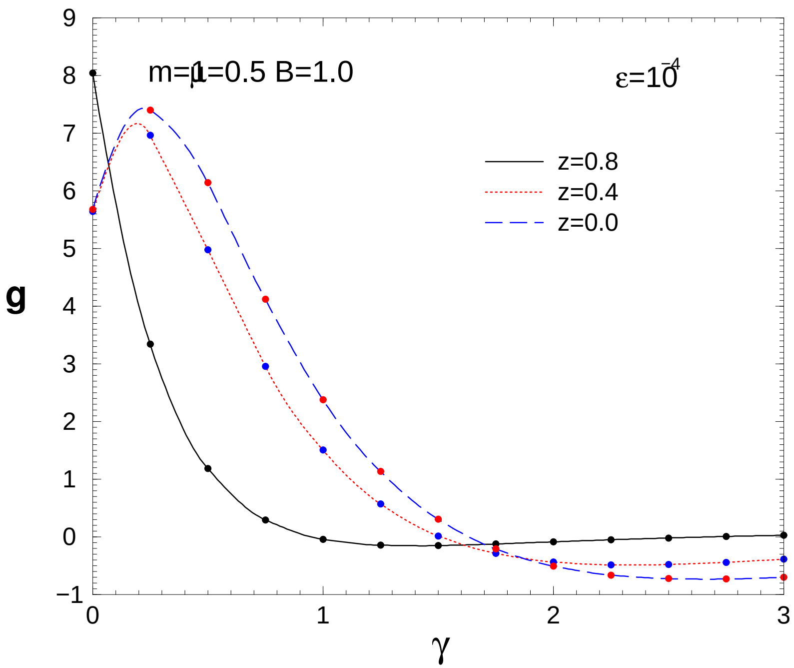
B	$\alpha(\mu = 0.15)$	$\alpha(\mu = 0.50)$
0.01	0.5716	1.440
0.10	1.437	2.498
0.20	2.100	3.251
0.50	3.611	4.901
1.00	5.315	6.712

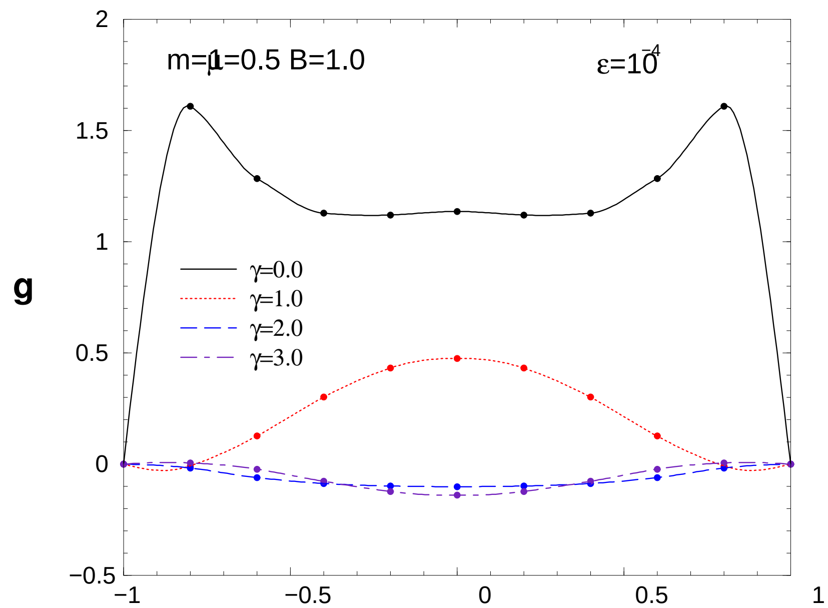
These results, with all shown digits, coincide with ones obtained in Euclidean space (by Wick rotation).

• This is a test of the method.

Function $g(\gamma, z)$

Function $g(\gamma, z)$ for $\mu = 0.5$ and $B = 1.0$.

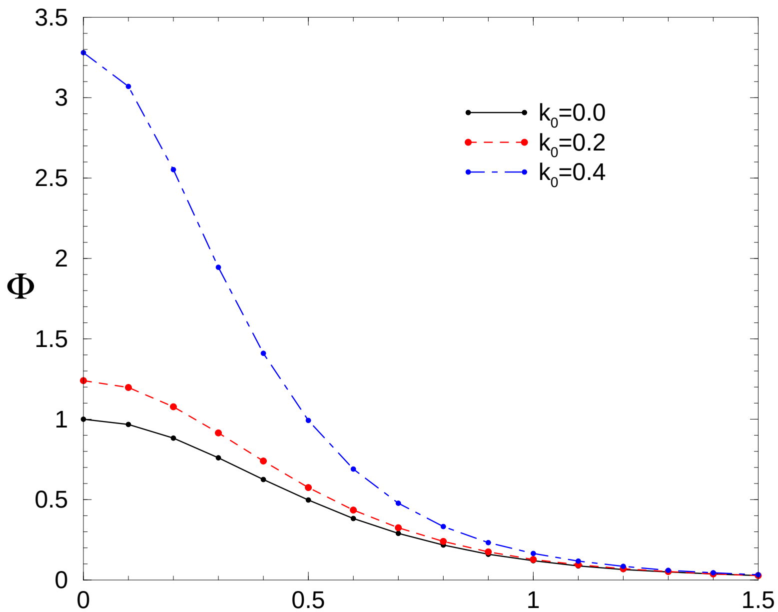


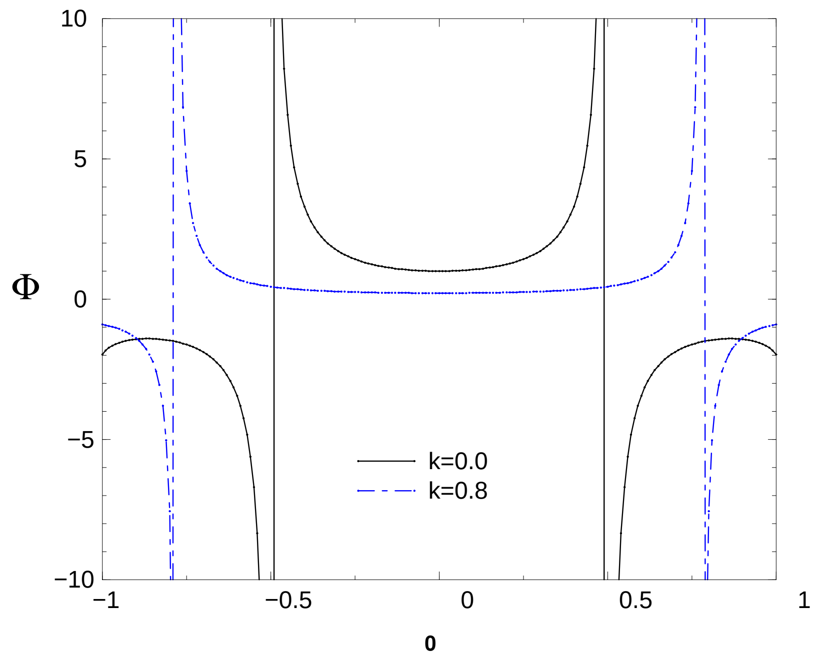


On left – versus γ for fixed values of z.
On right – versus z for a few fixed values of γ.

BS amplitude $\Phi(k_0, k) \quad (\vec{p} = 0)$

BS amplitude $\Phi(k_0, k)$ for $\mu = 0.5$ and $B = 1.0$.

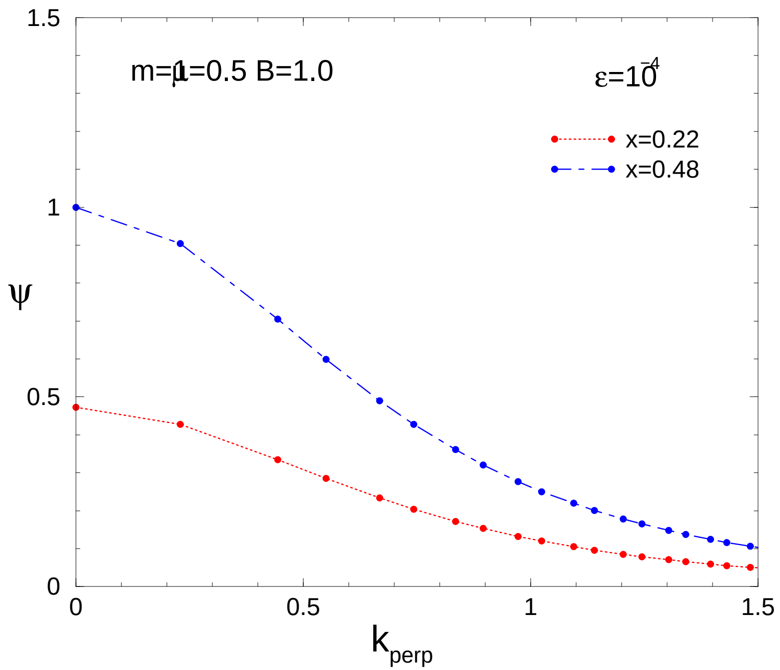


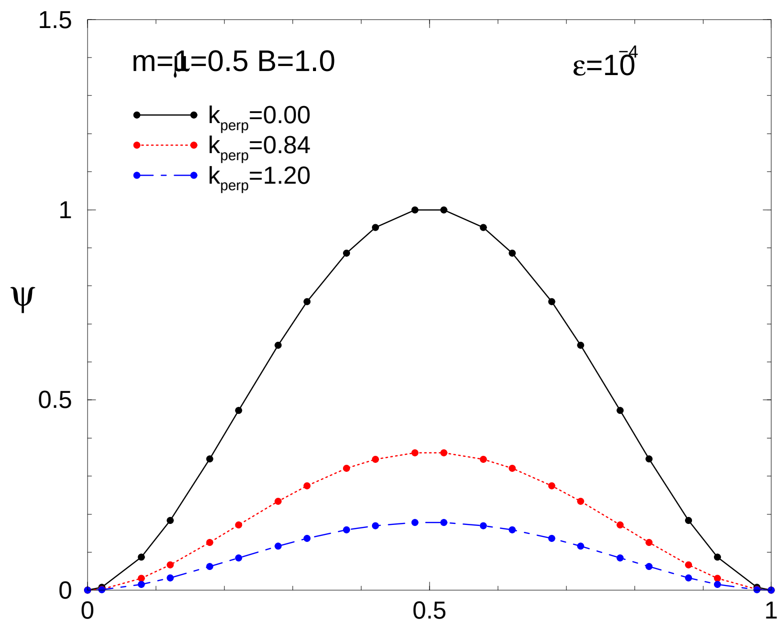


On left: versus k for fixed values of k_0.
On right: versus k_0 for a few fixed values of k.

LF wave function $\psi(k_\perp, x)$

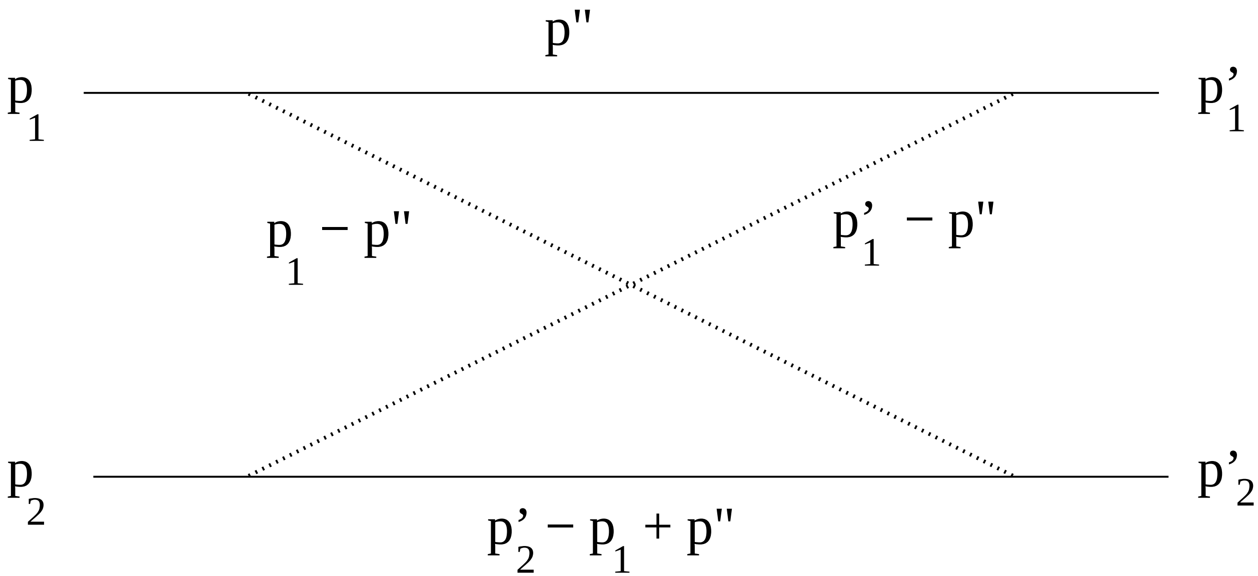
LF wave function $\psi(k_\perp, x)$ for $\mu = 0.5$ and $B = 1.0$.





On left: versus $k_\perp$ for fixed values of x
On right: versus x for a few fixed values of $k_\perp$.

• Feynman cross-box kernel



The BS equation in Minkowski space for the cross-box kernel has been also solved

(Carbonell, J., Karmanov, V.A., Eur. Phys. J. A **27**, 11 (2006))

• LFD approach

For comparison, we also solved LFD equation for ladder +cross ladders +stretched boxes.

$$\left(\frac{\vec{k}_\perp^2 + m^2}{x(1-x)} - M^2\right)\psi(\vec{k}_\perp, x)$$

$$= -\frac{m^2}{2\pi^3}\int \psi(\vec{k}'_\perp, x')V(\vec{k}'_\perp, x'; \vec{k}_\perp, x, M^2)\frac{d^2k'_\perp dx'}{2x'(1-x')}$$

$$V(\vec{k}'_\perp, x'; \vec{k}_\perp, x, M^2) = V^{(ladder)} + V^{(cr.ladder)} + V^{(str.box)}$$

$$V^{(cr.ladder)} = \sum_{i=1,\ldots,6} V_i$$

$$V^{(str.box)} = \sum_{i=7,8} V_i$$

• LF cross- and stretched boxes.

LFD cross ladders

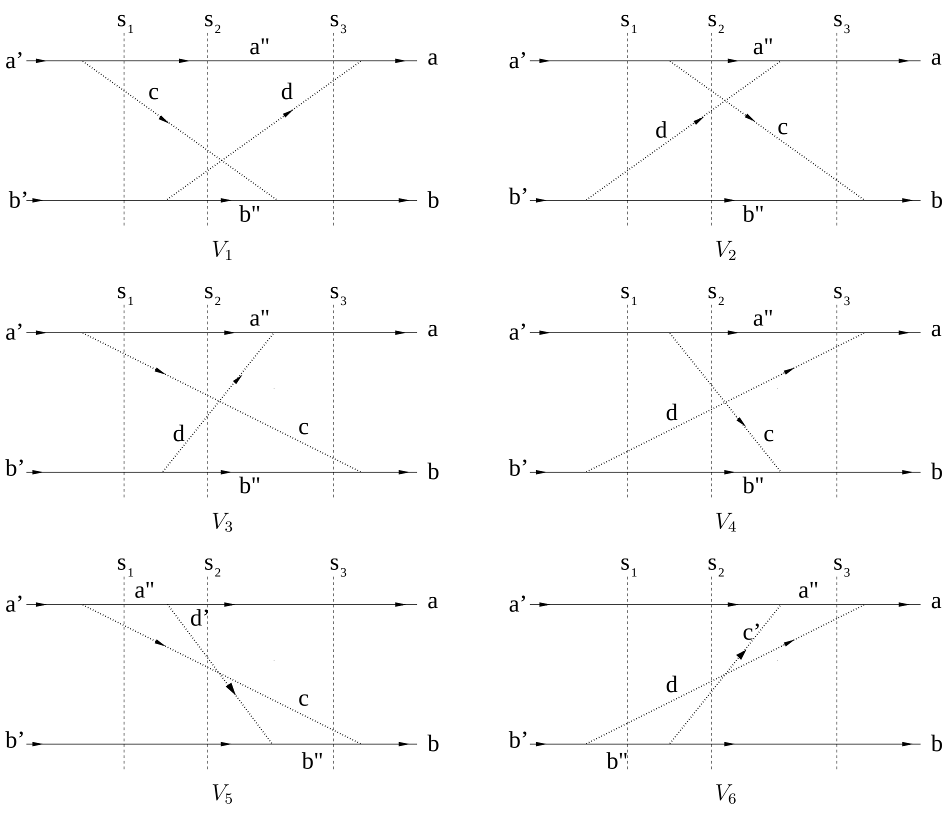
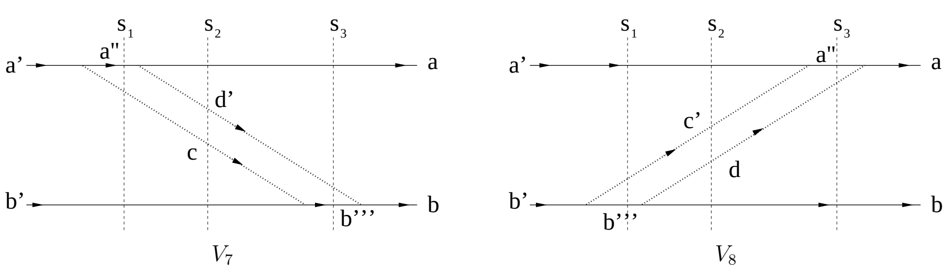
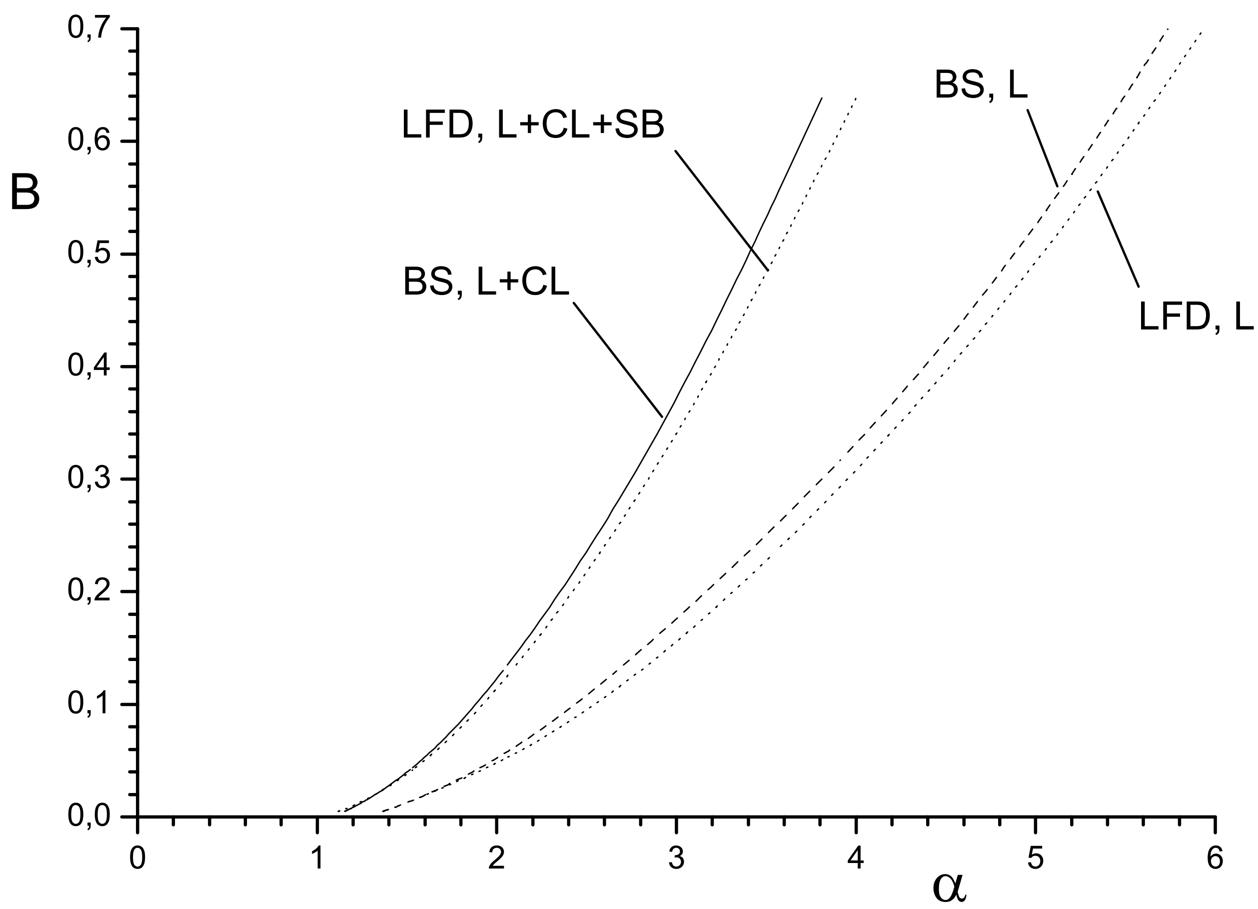


Figure 1: Cross LFD graphs.

LFD stretched boxes



• Numerical results for L +CL +SB



Binding energy B vs. coupling constant α for BS and LFD equations with the ladder (L) kernels only, with the ladder +cross-ladder (L+CL) and with the ladder +cross-ladder +stretched boxes (L+CL+SB) for exchange mass $\mu = 0.5$.

- BS and LFD binding energies are close to each other.
- Stretched box $V^{(str.box)}$ is small

  N.C.J. Schoonderwoerd, B.L.G. Bakker and V.A. Karmanov, Phys. Rev. **C 58** *(1998) 3093.*
- Cross ladder $V^{(cr.ladder)}$ is large

• Two fermions

This is much more realistic case.

BS amplitude can be decomposed in a spin basis

$(J^\pi = 0^+)$:

$$\Phi(k,p) = (S_1\phi_1 + S_2\phi_2 + S_3\phi_3 + S_4\phi_4)$$

where

$$S_1 = \gamma_5, \quad S_2 = \frac{1}{M}\hat{p}\,\gamma_5, \quad S_3 = \frac{k\cdot p}{M^3}\hat{p}\,\gamma_5 - \frac{1}{M}\hat{k}\,\gamma_5,$$
$$S_4 = \frac{i}{M^2}\sigma_{\mu\nu}p_\mu k_\nu\,\gamma_5$$

with $\hat{p} = p_\mu\gamma^\mu, \quad \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

Four scalar functions $\phi_{1-4}(k,p)$.

• Nakanishi representation

Nakanishi representation for all components ϕ_i.

$$\phi_i(k,p) = \frac{-i}{\sqrt{4\pi}}\int_{-1}^{1} dz' \times \int_0^{\infty} d\gamma' \frac{g_i(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\; z' - i\epsilon\right]^3}.$$

Sauli, V., J. Phys. G **35**, 035005 (2008) (for PS exchange);
Carbonell, J., Karmanov, V.A., Eur. Phys. J. A **46**, 387 (2010).

• System of equations

$$\int_0^\infty \frac{g_i(\gamma', z)d\gamma'}{\left[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2\right]^2} =$$

$$\sum_{j=1,2,3,4} \int_0^\infty d\gamma' \int_{-1}^{1} dz' \, V_{ij}(\gamma, z; \gamma', z') g_j(\gamma', z')$$

The 4×4 kernel matrix is calculated similarly to the spinless case.

• Meson exchange Lagrangians

Scalar meson exchange Lagrangian:

$$\mathcal{L}^{int} = g_s \, \bar{\psi}\psi\phi^{(s)}$$

Pseudoscalar meson exchange Lagrangian:

$$\mathcal{L}^{int} = i \, g_{ps} \, \bar{\psi}\gamma_5\psi \, \phi^{(ps)}$$

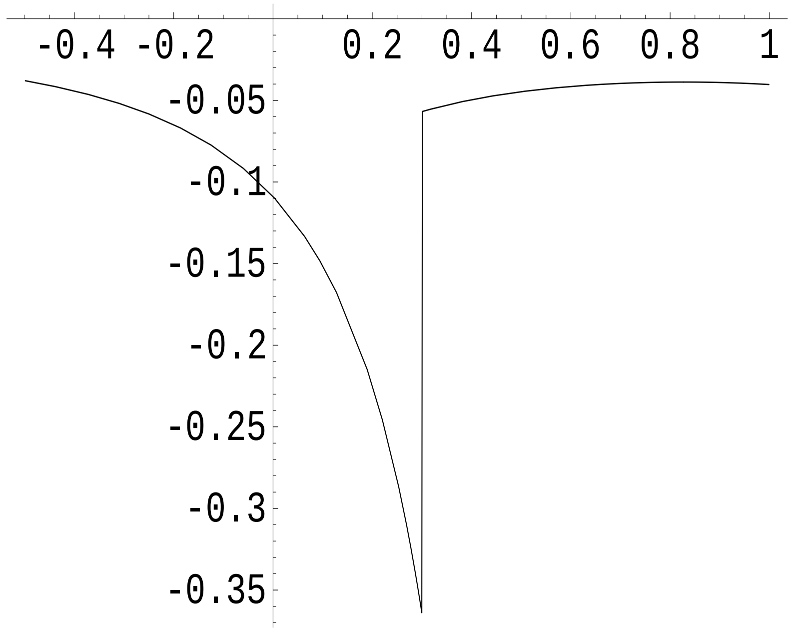
Massless vector exchange:

$$\mathcal{L}^{int} = g \, \bar{\Psi}\gamma^{\mu}V^{\mu} \, \Psi$$

with $\Pi_{\mu\nu} = -ig_{\mu\nu}/q^2$ as vector propagator.
Vertex form factor:

$$F(q) = \frac{\Lambda^2 - \mu^2}{\Lambda^2 - q^2}$$

• Discontinuity of $V_{ij}(\gamma, z; \gamma', z')$



One of the matrix elements V_{ij} at $z = 0.3$ v.s. z'. One can see the discontinuity at $z' = z$.

No catastrophe, but we should take care, choosing a method of the z' integration.

• Improving the method

- Take the BS equation and multiply both sides by $\eta(k,p)$:

$$\eta(k,p)\;\Phi(k,p) = \frac{-i\eta(k,p)}{\left((\frac{p}{2}+k)^2-m^2+i\epsilon\right)\left((\frac{p}{2}-k)^2-m^2+i\epsilon\right)}\int\frac{d^4k'}{(2\pi)^4}K(k,k',p)\Phi(k',p)$$

where

$$\begin{aligned}\eta(k,p) &= \frac{(m^2-L^2)}{(k_1^2-L^2+i\epsilon)}\frac{(m^2-L^2)}{(k_2^2-L^2+i\epsilon)}\\ &= \frac{(m^2-L^2)}{\left((\frac{p}{2}+k)^2-L^2+i\epsilon\right)}\frac{(m^2-L^2)}{\left((\frac{p}{2}-k)^2-L^2+i\epsilon\right)}\end{aligned}$$

Equation remains the same!

- Use Nakanishi representation and apply to both sides the LF projection $\int\ldots dk_-$.
- Obtain new equation for $g(\gamma,z)$. L appears in the equation, but the result does not depend on it!

• New equation for $g(\gamma, z)$

$$\int_0^\infty d\gamma' \int_{-1}^1 dz' F(\gamma, z; \gamma', z')\, g_i(\gamma', z') = \int_0^\infty d\gamma' \int_{-1}^1 dz' \sum_{ij} V_{ij}(\gamma, z; \gamma', z') g_j(\gamma', z')$$

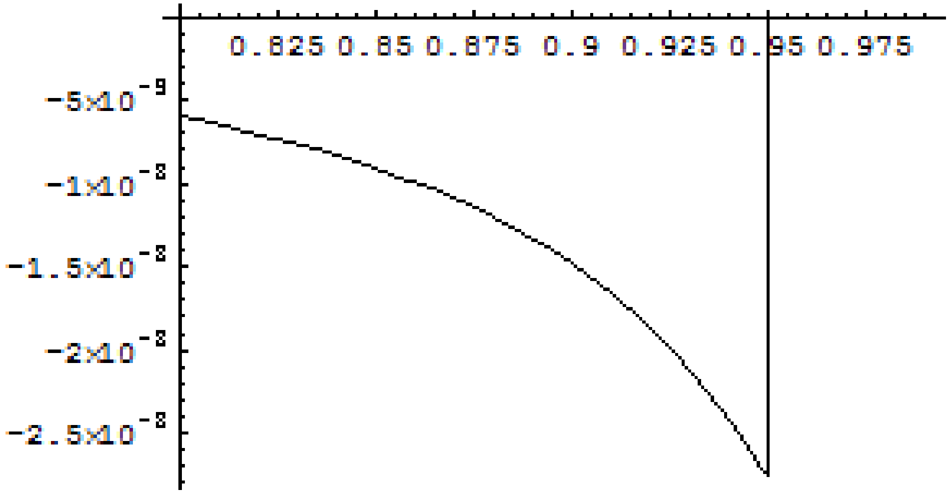
(Notice double integral in l.h.s.) Here

$$F(\gamma, z; \gamma', z') = \begin{cases} f(\gamma, z; \gamma', z'), & \text{if } -1 \le z' \le z \le 1 \\ f(\gamma, -z; \gamma', -z'), & \text{if } -1 \le z \le z' \le 1 \end{cases}$$

$$f(\gamma, z; \gamma', z') = \frac{(L^2 - m^2)}{\left[\gamma \frac{(1-z')}{(1-z)} + \gamma' + (1 - z')(1 + z)\kappa^2 + \Big(z' - z(1 - z')\Big)m^2 + \frac{(z-z')}{(1-z)} L^2\right]^3}$$

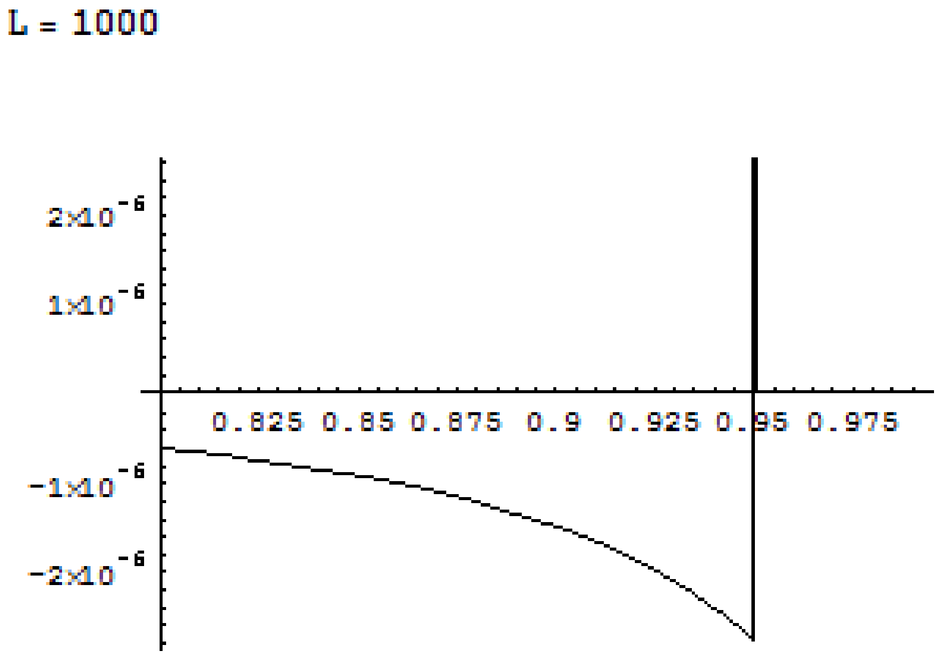
L.h.s. changes. The kernel in r.h.s. changes (becomes more smooth). The solution remain the same, though it is found much more easy.

Kernel $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z', $L = 10000$



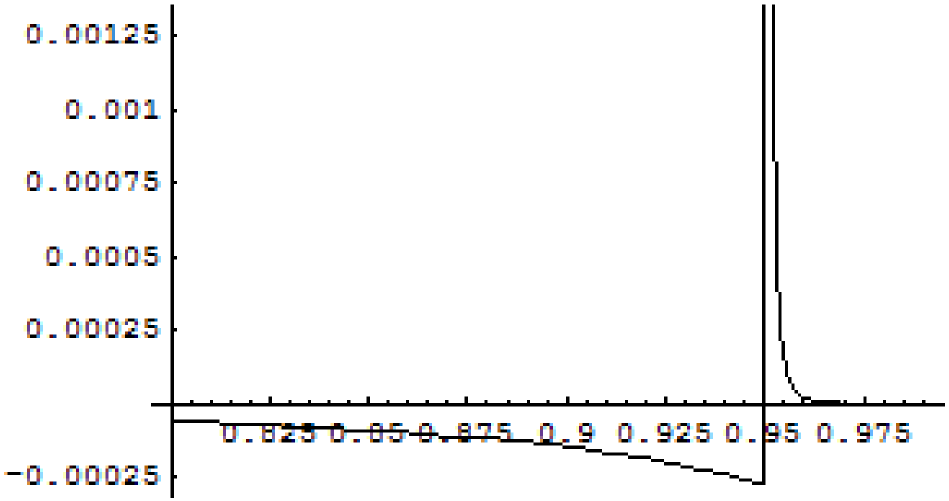
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 10000$

Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 1000$



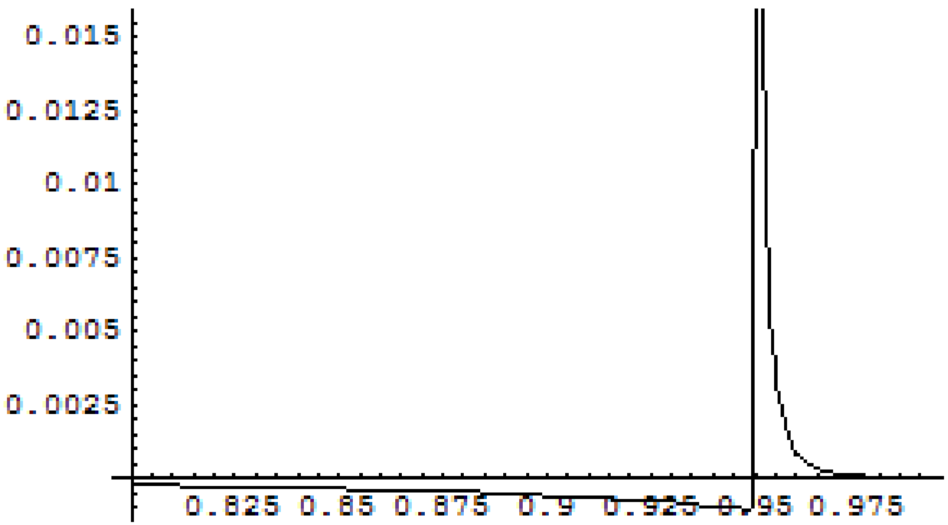
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 1000$

Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 100$



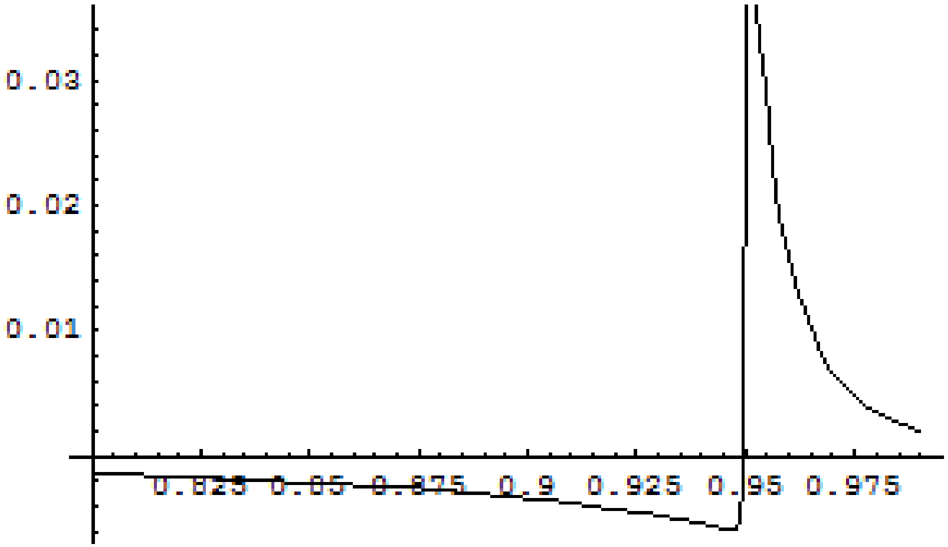
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 100$

• Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 50$



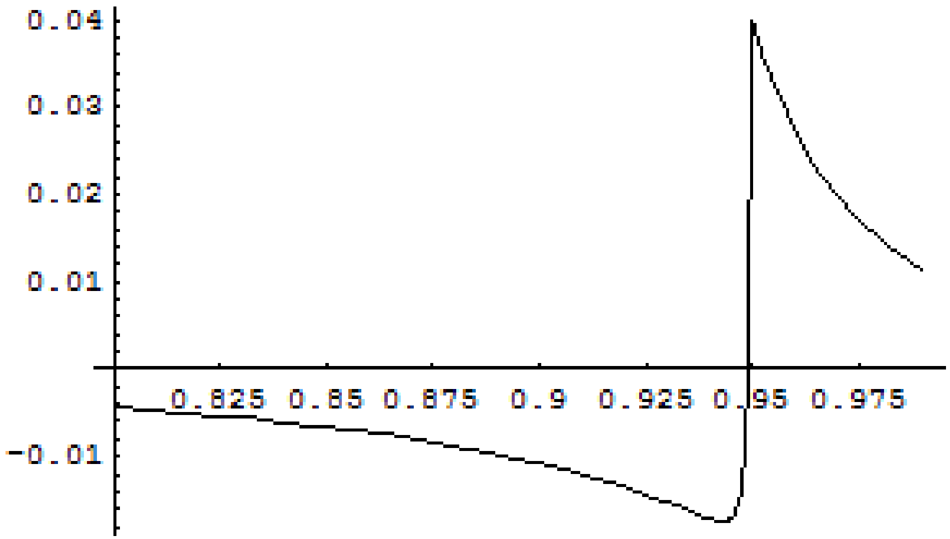
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 50$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 20$



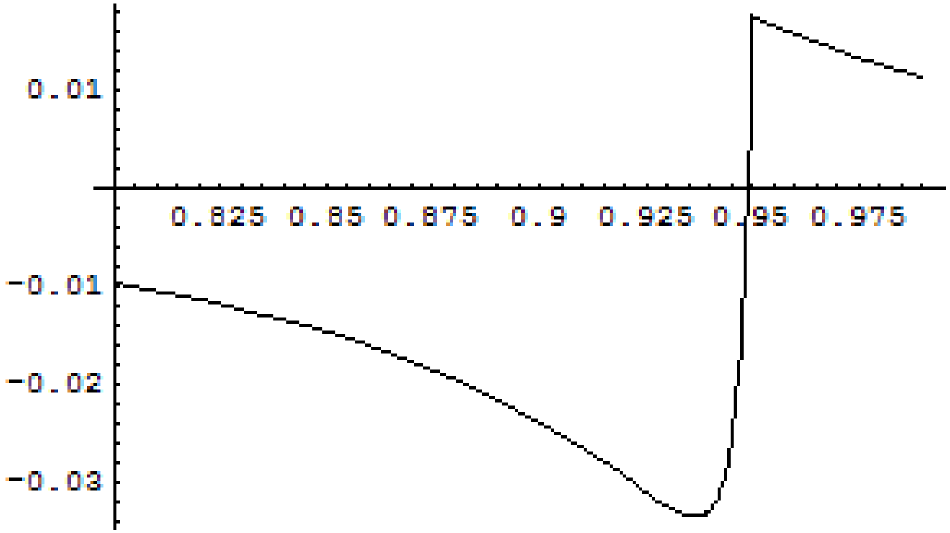
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 20$

- **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 10$



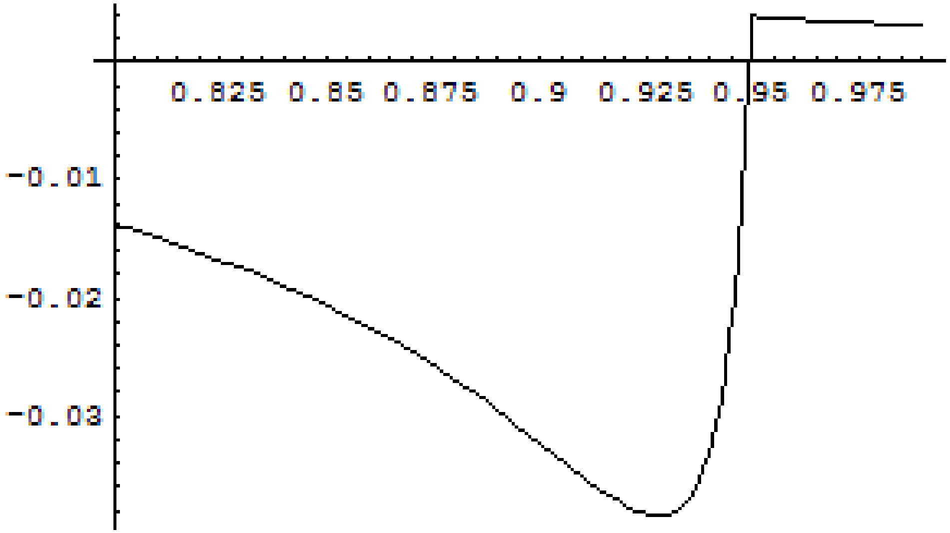
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 10$

• Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 5$



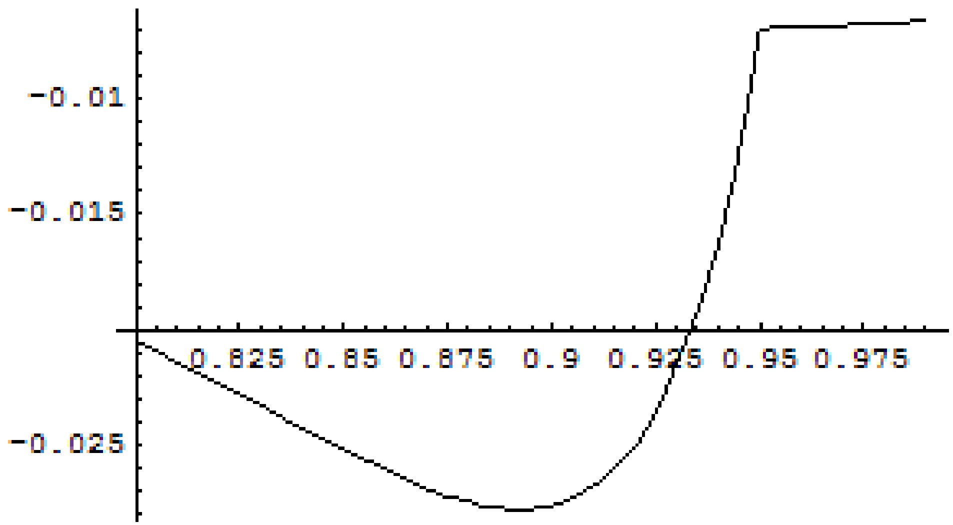
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 5$

• Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 3$



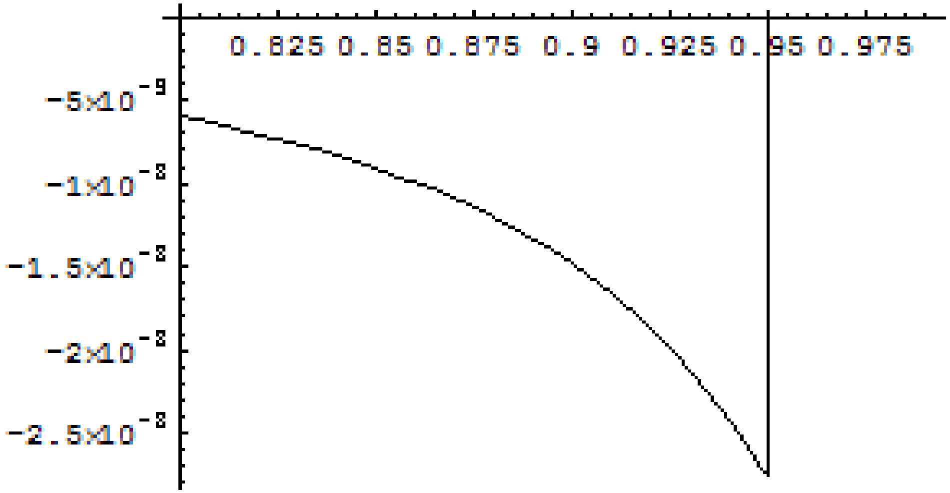
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 3$

• Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 1.1$



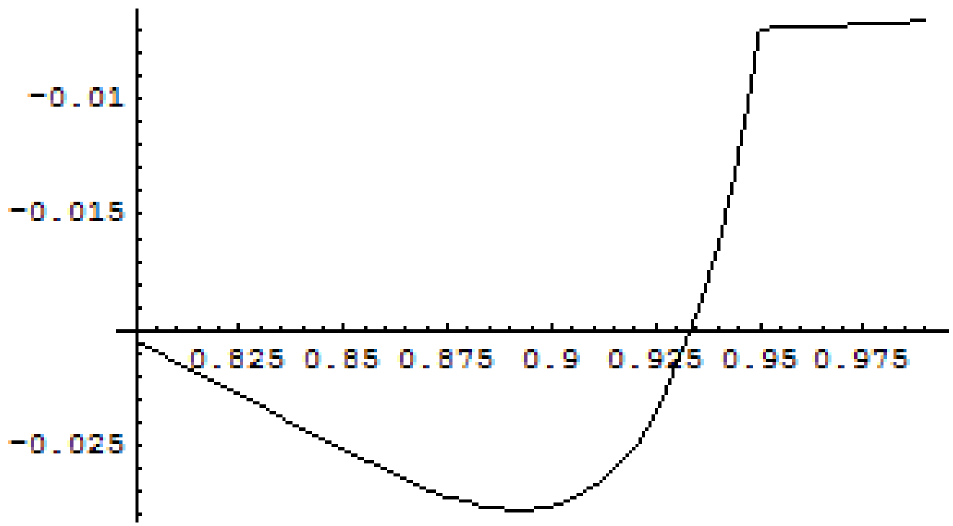
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 1.1$

Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 10000$



Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 10000$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 1.1$



Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 1.1$

• Numerical results

Scalar exchange (Yukawa model)

$\mu = 0.15,\ \Lambda = 2,\ L = 1.1$

B	g^2 (Dorkin et al.)	g^2 (C&K, Eucl.)	g^2 (C&K, Mink.)
0.08104	20.23	20.23	20.23
0.14773	30.34	30.34	30.34
0.27765	50.57	50.57	50.57

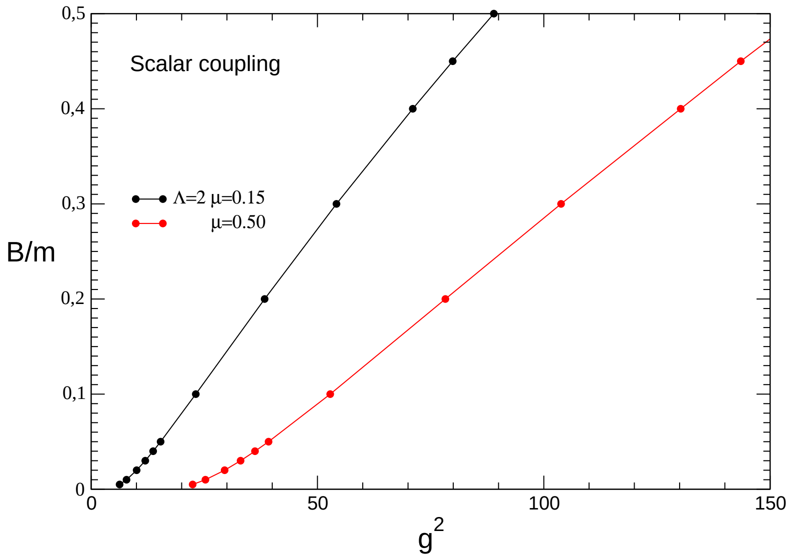
S.M. Dorkin, M. Beyer, S.S. Semykh and L.P. Kaptari, Few-Body Systems, **42**, 1, (2008).

J. Carbonell, V.A. Karmanov, Eur. Phys. J. A **46**, 387 (2010).

Binding energies, found via Minkowski and Euclid, coincide now within 4 digits.

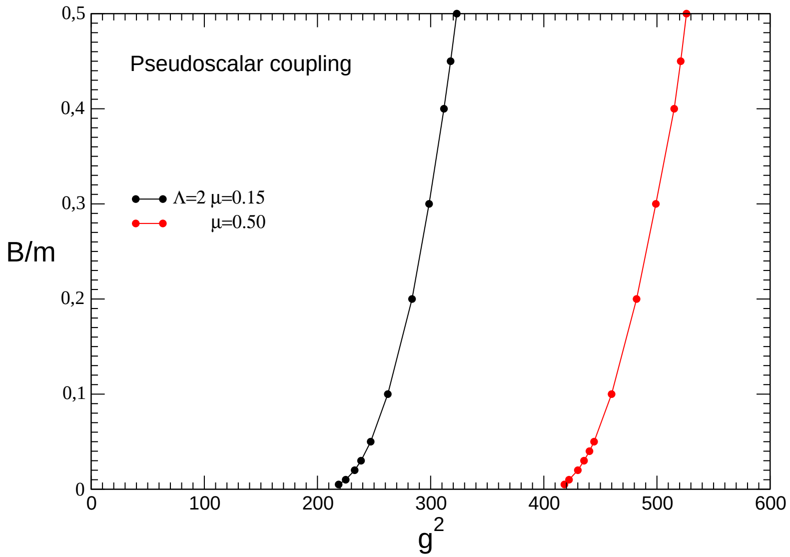
Good precision!

• Binding energy for scalar exchange



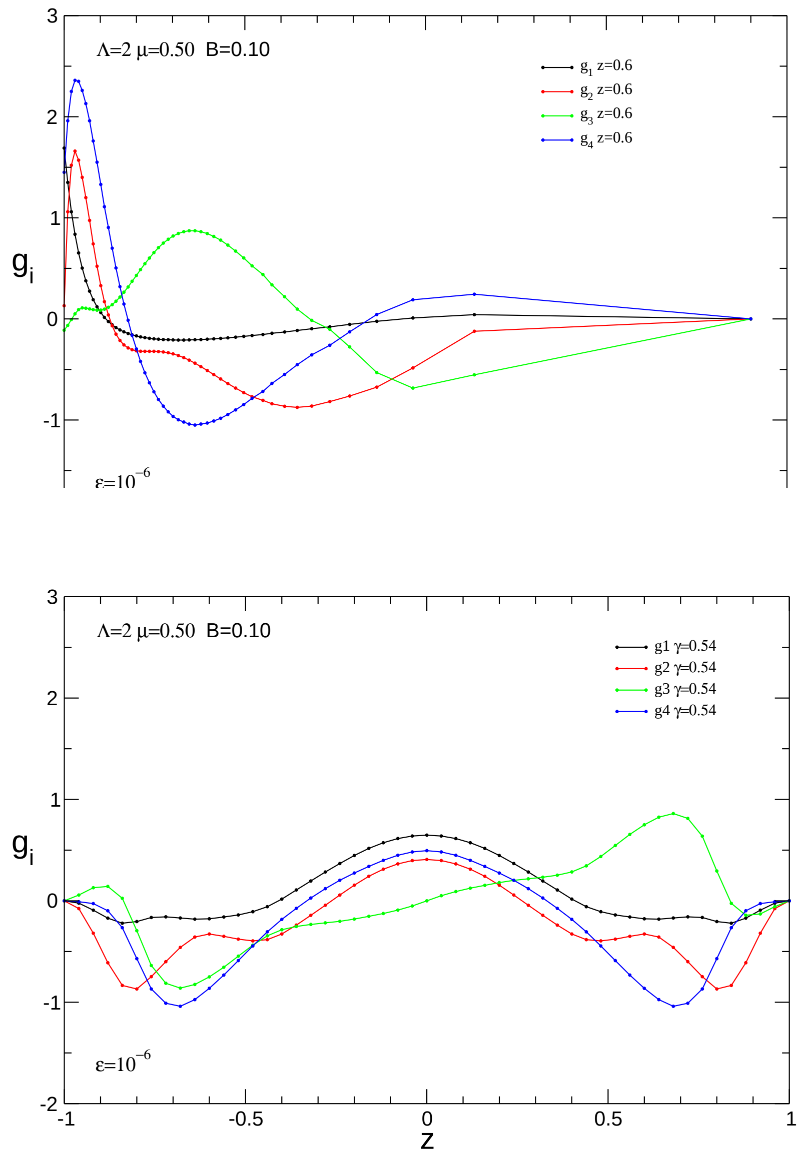
Binding energy for scalar exchange v.s. g^2 for $\Lambda = 2$, $L = 1.1$, $\mu = 0.15$ and $\mu = 0.5$

Binding energy for pseudoscalar exch.

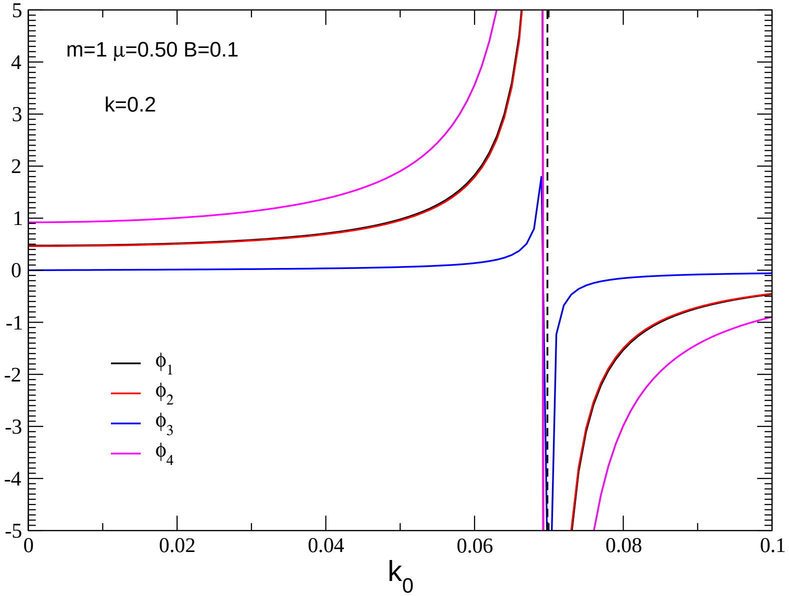


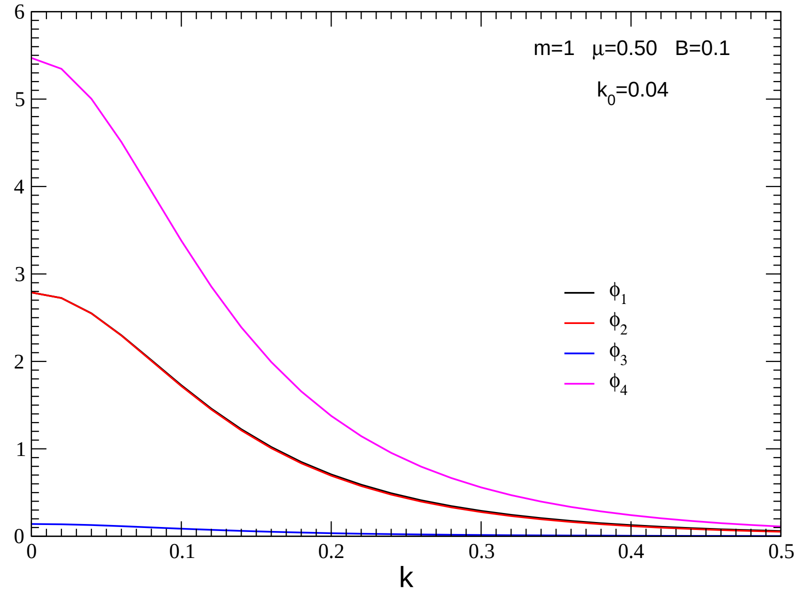
Binding energy for pseudo scalar exchange v.s. g^2 for $\Lambda = 2$, $L = 1.1$, $\mu = 0.15$ and $\mu = 0.5$

• Nakanishi functions $g(\gamma, z)$



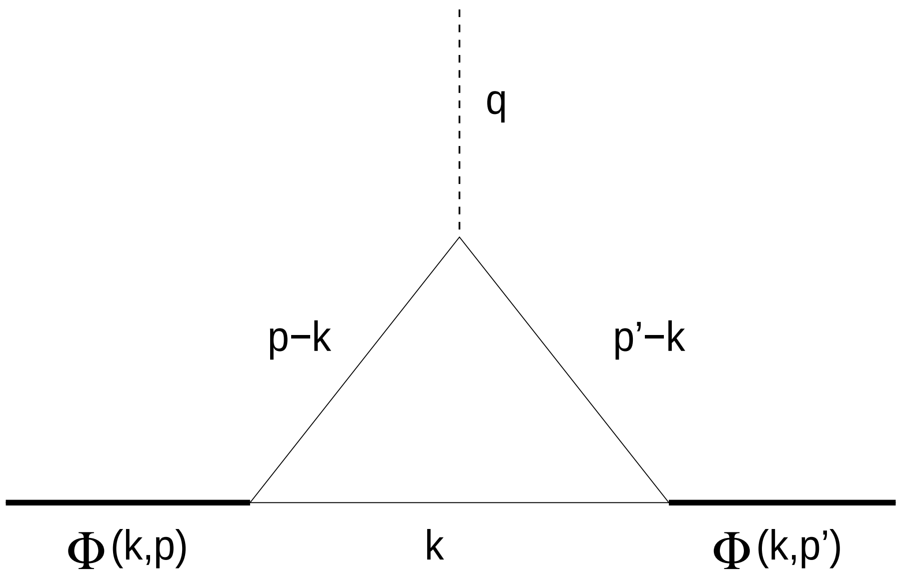
• BS amplitude in Minkowski space





Bethe-Salpeter Minkowski amplitudes, for scalar exchange, v.s. k_0 for $k = |\vec{k}| = 0.2$ (on left) and v.s. $k = |\vec{k}|$ for $k_0 = 0.04$ (on right). The amplitudes ϕ_1 and ϕ_2 are indistinguishable.

• E.M. form factor



E.m. vertex in terms of the BS amplitude.

$$(p+p')^\mu F_M(Q^2) = -i \int \frac{d^4k}{(2\pi)^4}(p+p'-2k)^\mu \ (m^2-k^2)\Phi_M\left(\frac{p}{2}-k,p\right)\Phi_M\left(\frac{p'}{2}-k,p'\right)$$

• E.M. form factor via $g(\gamma, z)$

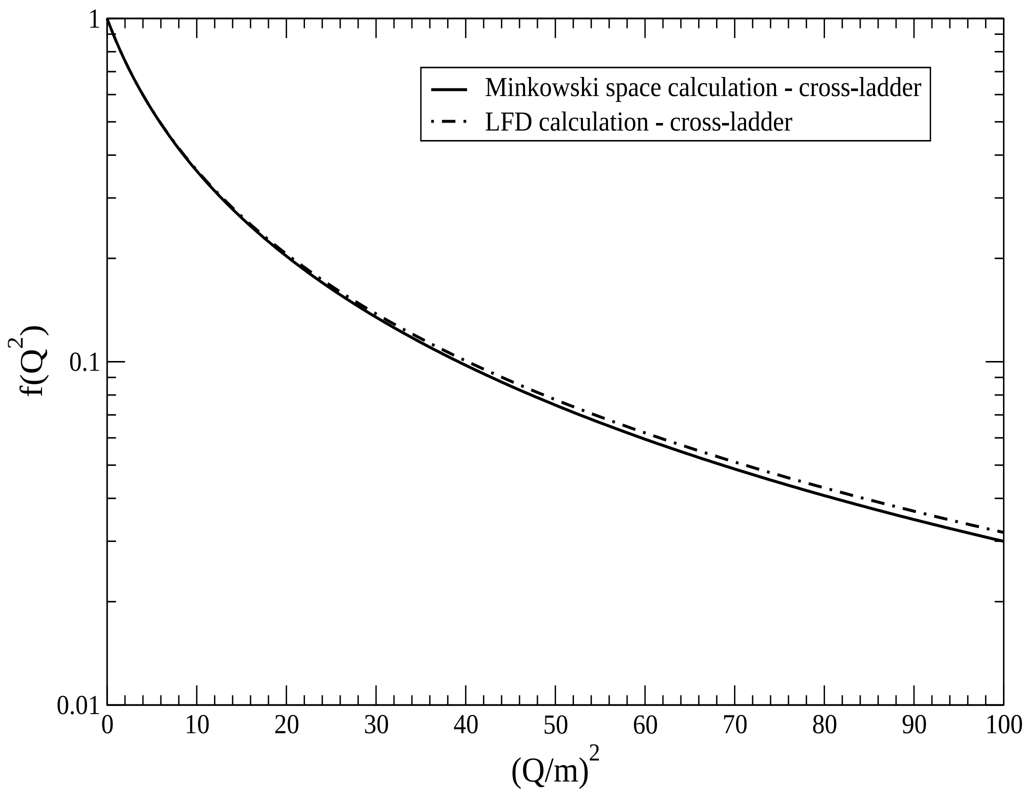
$$F_M(Q^2) = \frac{1}{2^7 \pi^3 N_M} \int_0^\infty d\gamma \int_{-1}^1 dz\, g(\gamma, z)$$
$$\times \int_0^\infty d\gamma' \int_{-1}^1 dz'\, g(\gamma', z') \int_0^1 du\, u^2 (1-u)^2 \frac{f_{num}}{f_{den}^4}$$

with

$$\begin{aligned} f_{num} &= (6\xi - 5)m^2 + [\gamma'(1-u) + \gamma u](3\xi - 2) \\ &+ 2M^2 \xi(1-\xi) + \frac{1}{4} Q^2 (1-u)u(1+z)(1+z') \\ f_{den} &= m^2 + \gamma'(1-u) + \gamma u - M^2(1-\xi)\xi \\ &+ \frac{1}{4} Q^2 (1-u)u(1+z)(1+z'), \end{aligned}$$

$$\xi = \frac{1}{2}(1+z)u + \frac{1}{2}(1+z')(1-u).$$

• E.M. form factor, numerical



Form factor via Minkowski BS amplitude (solid curve), and in LFD (dot-dashed curve).

• BS equation for the scattering states

$$F(k,k'',P) = V^{inh}(k,k'',P) - i\int \frac{d^4k'}{(2\pi)^4} \times \frac{V(k,k',P)F(k',k'',P)}{\left[\left(\frac{1}{2}P+k'\right)^2 - m^2 + i\epsilon\right]\left[\left(\frac{1}{2}P-k'\right)^2 - m^2 + i\epsilon\right]}$$

Ladder kernel: $V(k,k',P) = -\frac{g^2}{(k-k')^2 - \mu^2 + i\epsilon}$

k – relative 4-momentum (variable).
k' – integration 4-momentum (variable).
k'' – physical (relative) 4-momentum.
P – total 4-momentum.

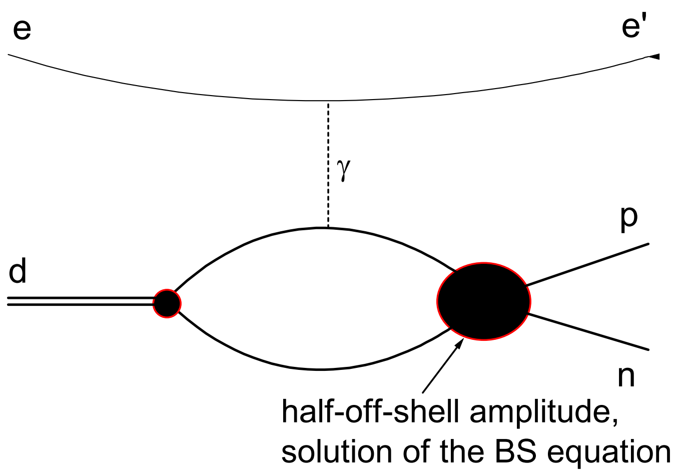
It determines the off-shell amplitude in Minkowski space.

In c.m.-frame $\vec{P}=0$: $F = F(k_0, k; k'')$ depends on two variables k_0, k (for S-wave).

On-mass shell: $F^{on} = F(k_0 = 0, k = k''; k'') = F(k'')$ – physical amplitude.

• Scattering states

- On-mass shell amplitude $\rightarrow$ phase shifts.
- Off-mass shell amplitude $\rightarrow$ transition form factors. And also as an input for the three-body BS-Faddeev equations.



Deutron electrodisintrgrastion $ed \rightarrow enp$

• On-shell scattering BS amplitude

Phase shifts via BS were found more than 40 years ago:

M.J. Levine, J.A. Tjon, J. Wright, (1966);

C. Schwartz, C. Zemach, (1966);

P. Graves-Morris, (1966);

R. Haymaker, (1967).

• Fnding off-shell BS amplitude

- Separable kernel.
  Phase shifts $\to$ fitting kernel $\to$ off-shell amplitude.
  S.S. Bondarenko, V.V. Burov, E.P. Rogochaya,
  Phys. Lett. B **705**, 264 (2011).
- Nakanishi integral representation.
  T. Frederico, G. Salmè, and M. Viviani,
  Phys. Rev. D 85 (2012) 036009.
  As I am aware, full numerical solution was not yet obtained.
  However, the scattering length was calculated.
- Direct treating of singularities.
  V.A. Karmanov and J. Carbonell,
  FB20, Fukuoka, Japan (2012); arXiv:1210.0925;
  Proc. of Sci. (Baldin ISHEPP XXI)027 (2012).

• Nakanishi for scattering

(T. Frederico, G. Salmè, and M. Viviani, Phys. Rev. D 85 (2012) 036009.)

$$F(k,k',P) = -i\int_{-1}^{1} dz' \int_{-1}^{1} dz'' \int_{-\infty}^{\infty} d\gamma' \frac{g(z',z'',\gamma')}{[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\, z'' - 2k\cdot k'\, z' - i\epsilon]^3}$$

Advantages

- No partial wave decomposition.
  All the partial waves are included!
- Form factors are expressed analytically via non-singular $g(z',z'',\gamma')$.

As I am aware, the numerical solution for $g(z',z'',\gamma')$ is not yet found.

• Direct treating of singularities

(J. Carbonell and V.A. Karmanov)

Four sources of singularities

1. Constituent propagators

$$\frac{1}{\left[\left(\frac{1}{2}P+p'\right)^2-m^2+i\epsilon\right]\left[\left(\frac{1}{2}P-p'\right)^2-m^2+i\epsilon\right]}$$

2. Kernel

$$V(p,p',P)=-\frac{g^2}{(p-p')^2-\mu^2+i\epsilon}$$

3. Inhomogeneous term

$$V^{inh}(p,p'',P)=-\frac{g^2}{(p-p'')^2-\mu^2+i\epsilon}$$

4. Amplitude $F(p',p'',P)$ itself

Product of four pole terms

$$\frac{1}{(p_0' - p_0^- - i\epsilon)} = PV \frac{1}{(p_0' - p_0^-)} + i\pi\delta(p_0' - p_0^-)$$

etc. (for all four products).

$$\int dp' dp_0' \,(PV + i\delta)\,(PV + i\delta)\,(PV + i\delta)\,(PV + i\delta)$$

$$= \int dp' dp_0' PV \cdot PV \cdot PV \cdot PV$$ ⇐ **2D integral**

$$+ \int dp' dp_0' PV \cdot PV \cdot PV \cdot \delta + \ldots$$ ⇐ **1D integral**

$$+ \int dp' dp_0' PV \cdot PV \cdot \delta \cdot \delta$$ ⇐ **0D integral**

• PV integral

$$PV \int_0^\infty \frac{dp_0}{p_0^2 - a^2} = 0$$

$$a = p_0^{\pm} = \varepsilon_{p'} \pm \varepsilon_{p''}$$

• BS equation for the S-wave amplitude

$$F_0(p_0,p;p'') = V_0^{inh}(p_0,p;p'')$$

$$+\int_0^\infty \frac{dp'}{\varepsilon_{p'}}\left\{\frac{i}{4\varepsilon_{p''}}\int_0^\infty \frac{dp'_0}{(p'^2_0 - p_0^{-\,2})}\left[V_0^s(p_0,p;p'_0,p')F_0(p'_0,p';p'')\right.\right.$$

$$\left. -V_0^s(p_0,p;p_0^-,p')F_0(|p_0^-|,p';p'')\right]$$

$$-\frac{i}{4\varepsilon_{p''}}\int_0^\infty \frac{dp'_0}{(p'^2_0 - p_0^{+\,2})}\left[V_0^s(p_0,p;p'_0,p')F_0(p'_0,p';p'')\right.$$

$$\left.\left. -V_0^s(p_0,p;p_0^+,p')F_0(p_0^+,p';p'')\right]\right\} \quad \Leftarrow \textbf{2D integral}$$

$$+\int_0^\infty \frac{dp'}{\varepsilon_{p'}}\left\{\frac{\pi}{8\varepsilon_{p''}}\frac{1}{(\varepsilon_{p'}-\varepsilon_{p''})}\left[V_0^s(p_0,p;p_0^-,p')F_0(|p_0^-|,p';p'')\right.\right.$$

$$\left. - \quad \frac{2\varepsilon_{p'}}{(\varepsilon_{p'}+\varepsilon_{p''})}V_0^s(p_0,p;p'_0=0,p'=p'')F_0(p'_0=0,p'=p'';p'')\right]$$

$$\left. -\frac{\pi}{8\varepsilon_{p''}}\frac{1}{(\varepsilon_{p'}+\varepsilon_{p''})}V_0^s(p_0,p;p_0^+,p')F_0(p_0^+,p';p'')\right\} \quad \Leftarrow \textbf{1D integral}$$

$$+\frac{i\pi^2}{8p''\varepsilon_{p''}}V_0^s(p_0,p;p'_0=0,p'=p'')F_l(p'_0=0,p'=p'';p'') \quad \Leftarrow \textbf{0D}$$

• 2. S-wave kernel

$$
\begin{aligned}
V_0(p_0, p, p'_0, p') &= -\int_{-1}^{1} \frac{g^2\, du}{(p_0 - p'_0)^2 - (p^2 - 2p\, p'\, u + p'^2) - \mu^2 + i\epsilon} \\
&= -\frac{8\pi m^2 \alpha}{pp'} \int_{-1}^{1} \frac{du}{\eta + u + i\epsilon} \\
&= -\frac{8\pi \alpha m^2}{pp'} \log \frac{|\eta + 1|}{|\eta - 1|} + \frac{i 8\pi \alpha m^2}{pp'} U(\eta)
\end{aligned}
$$

where

$$
\alpha = \frac{g^2}{16\pi m^2}, \quad \eta = \frac{(p_0 - p'_0)^2 - p^2 - p'^2 - \mu^2}{2pp'}
$$

and

$$
U(\eta) = \begin{cases} 1, & \text{if } |\eta| \leq 1 \\ 0, & \text{if } |\eta| > 1 \end{cases}
$$

Singularities of kernel

Kernel is singular when $\eta = \pm 1$. That is:

$$(p_0 - p'_0)^2 - (p \mp p')^2 - \mu^2 = 0$$

–Moving singularities in $p'_0 = p'_0(p')$.

In addition: 4 quadratic equations $\to$ 8 singularities.

$$(p_0 - p_0^{\pm})^2 - (p \pm p')^2 - \mu^2 = 0, \quad p_0^{\pm} = \varepsilon_{p'} \pm \varepsilon_{p''}$$

–Fixed singularities in p'.

All of them are $\log$-singularities.

To improve precision, we integrate numerically from one singularity to other.

$$\int_0^\infty \ldots dp' = \int_0^{p_1^{sing}} \ldots dp' + \int_{p_1^{sing}}^{p_2^{sing}} \ldots dp' + \int_{p_2^{sing}}^{p_3^{sing}} \ldots dp' + \ldots$$

In this integration we use appropriate change of variables.

• 3. Inhomogeneous term V^{inh}

The pole singularity which becomes the log-singularity for the partial wave.

• 4. Amplitude F itself

$$F = V^{inh} + V\Pi V^{inh} + V\Pi V^{inh}\Pi V^{inh} + \ldots$$

Amplitude F contains the singularities of each its iterative term.

The most dangerous ones result from the inhomogeneous term.

Introduce new function f:

$$F = \gamma V^{inh} f$$

γ is an arbitrary smooth function.

Inhomogeneous term in the equation for f is smooth.

$\Rightarrow$ f is now also smooth.

We solve equation for f.

The equation is lengthy,

but now the integrand is smooth, the integrals are easy computed!

We find numerical solution, decomposing it in the spline basis.

• Results for bound states

We reproduce the binding energies found previously, by other methods.

$$
\begin{aligned}
& B(\text{Minkowski space, present solution}) \\
= \; & B(\text{Nakanishi representation}) \\
= \; & B(\text{Euclidean space})
\end{aligned}
$$

The method works for the bound states!

• Extracting phase shift

$$F^{on} = F_l(p_0 = 0, p = p''; p'') \qquad \Leftarrow \text{on-mass shell}$$

$$S_l = e^{i2\delta_l} = 1 + \frac{2ip'' F^{on}}{\varepsilon_{p''}}$$

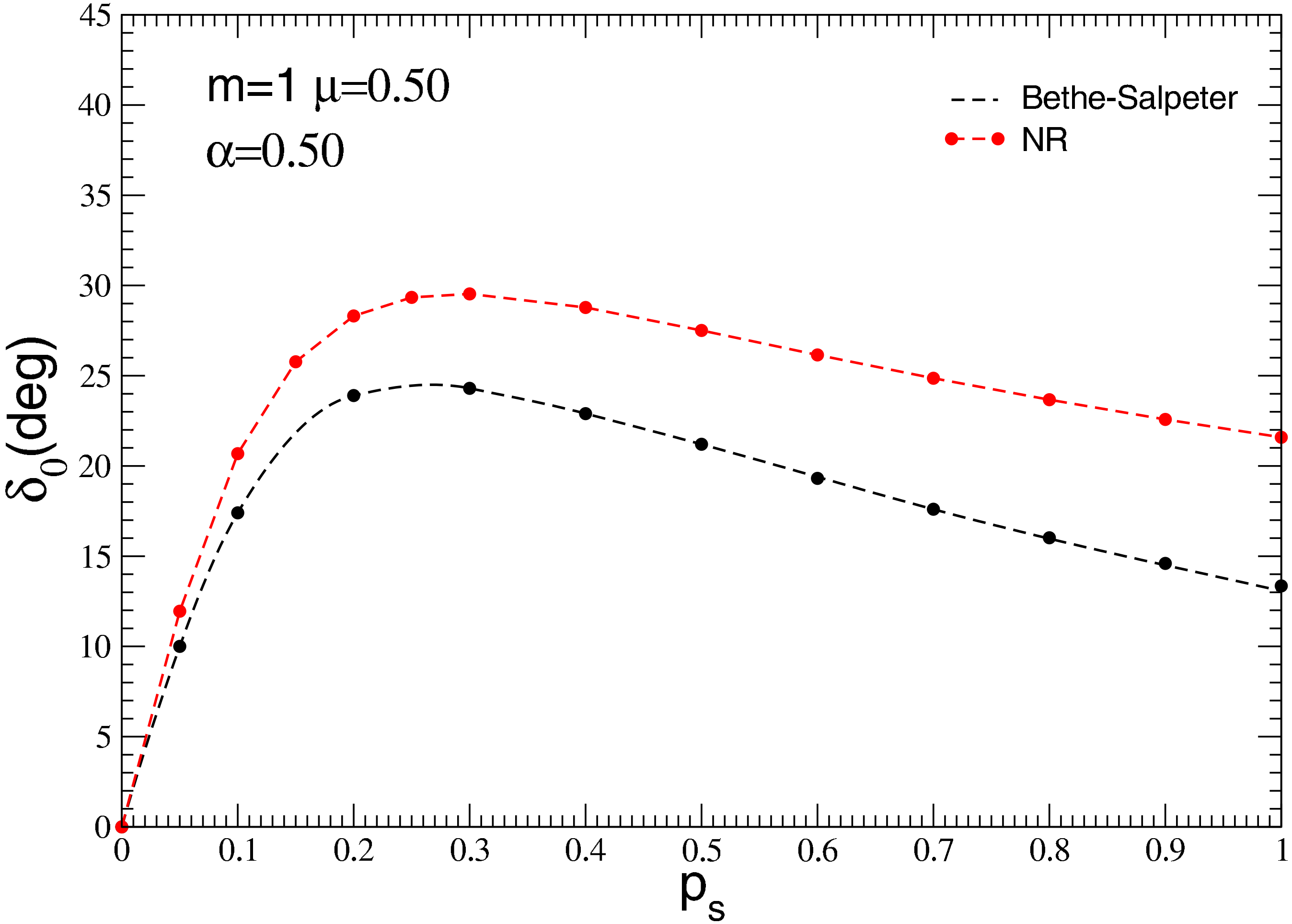
Or:

$$\delta_l = \frac{1}{2i} \log \left(1 + \frac{2ip'' F^{on}}{\varepsilon_{p''}} \right)$$

If $s < (2m + \mu)^2$, δ_l must be real. That is $|S_l| = 1$.

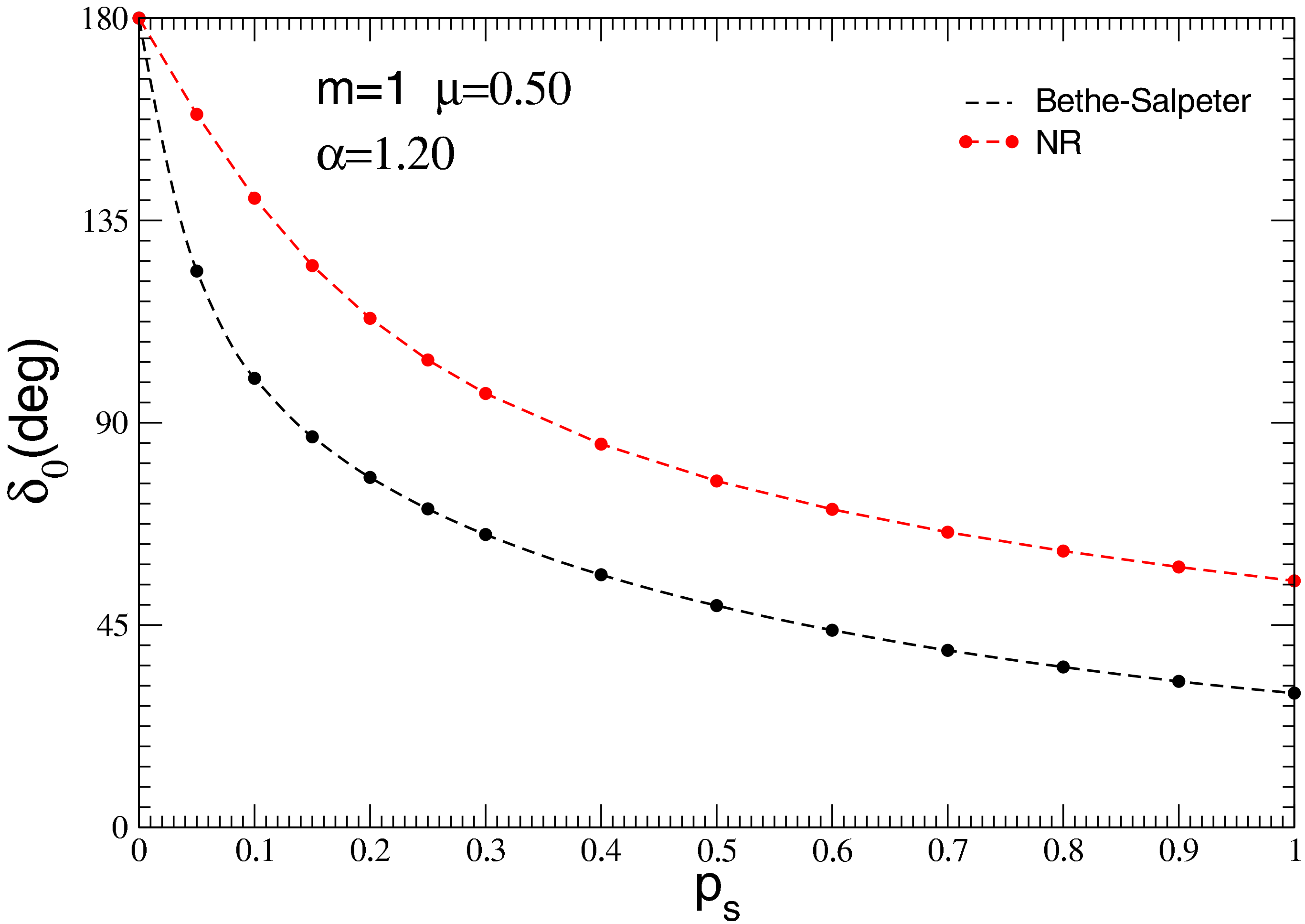
We check that our solution provides the phase shifts coinciding with ones found via the Euclidean space solution.

• Phase shift for $\alpha = \frac{g^2}{16\pi m^2} = 0.5$



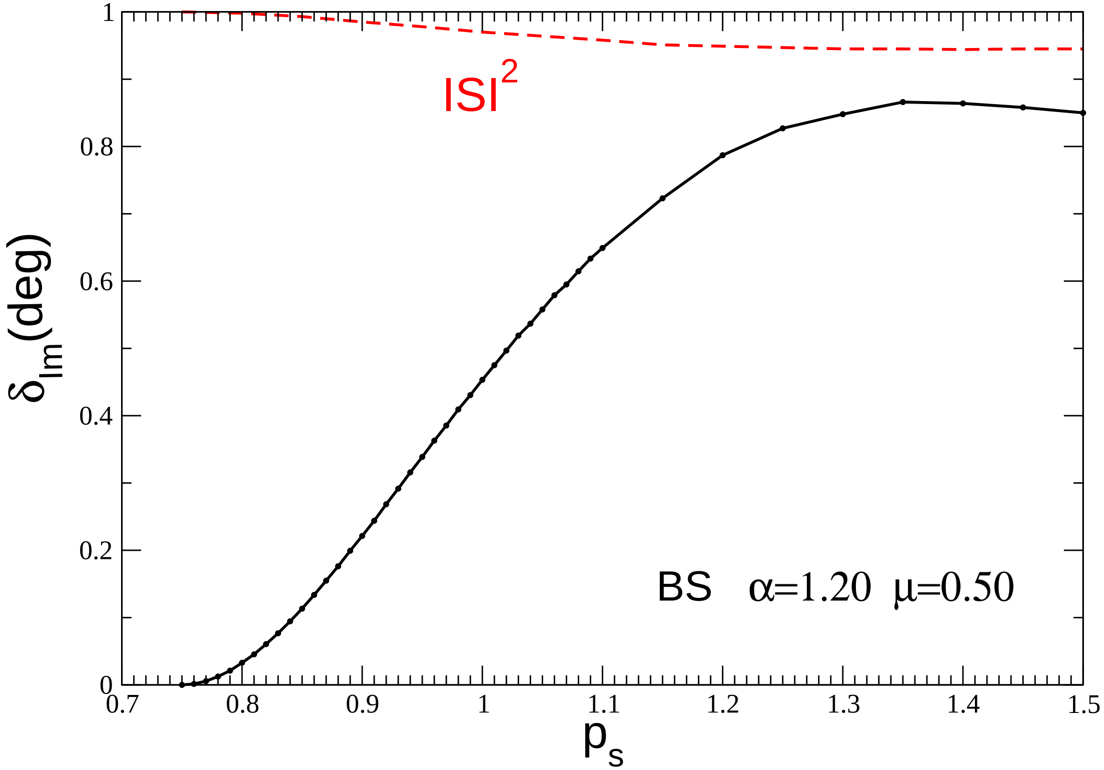
Precision is better than 0.1%.

• Phase shift for $\alpha = 1.2$ (bound state)



Precision is better than 0.1%.

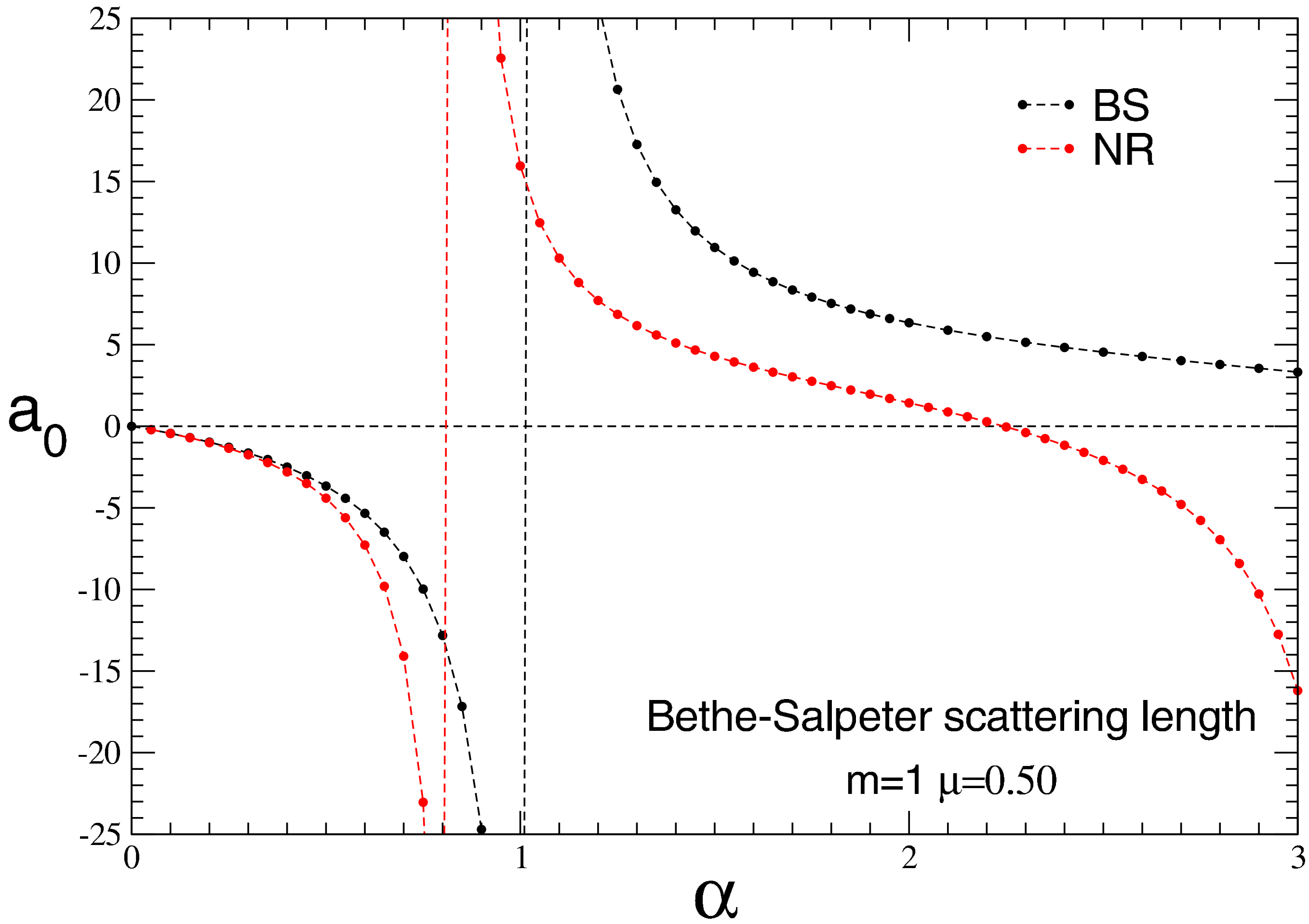
• Inelasticity



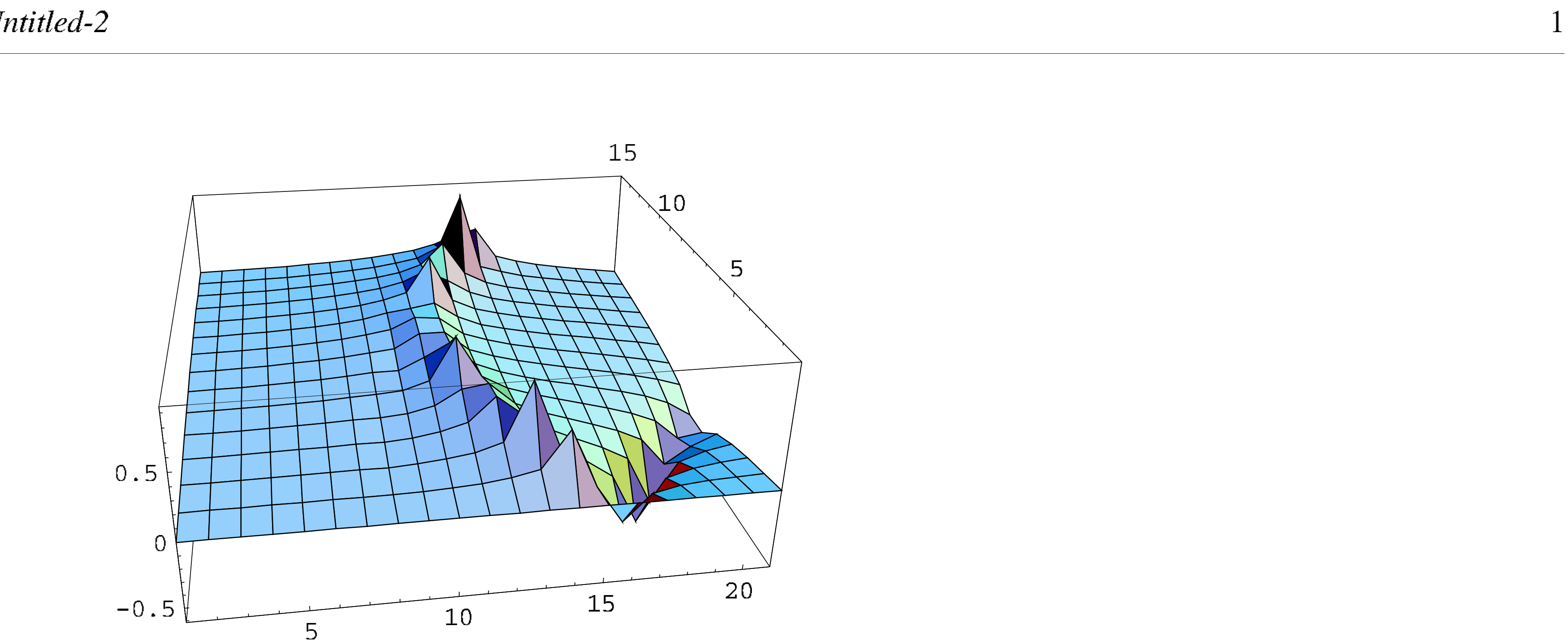
Imaginary part of δ

$$\mu = 0.5 \rightarrow p''_{threshold} = 0.75$$

• Scattering length vs. α

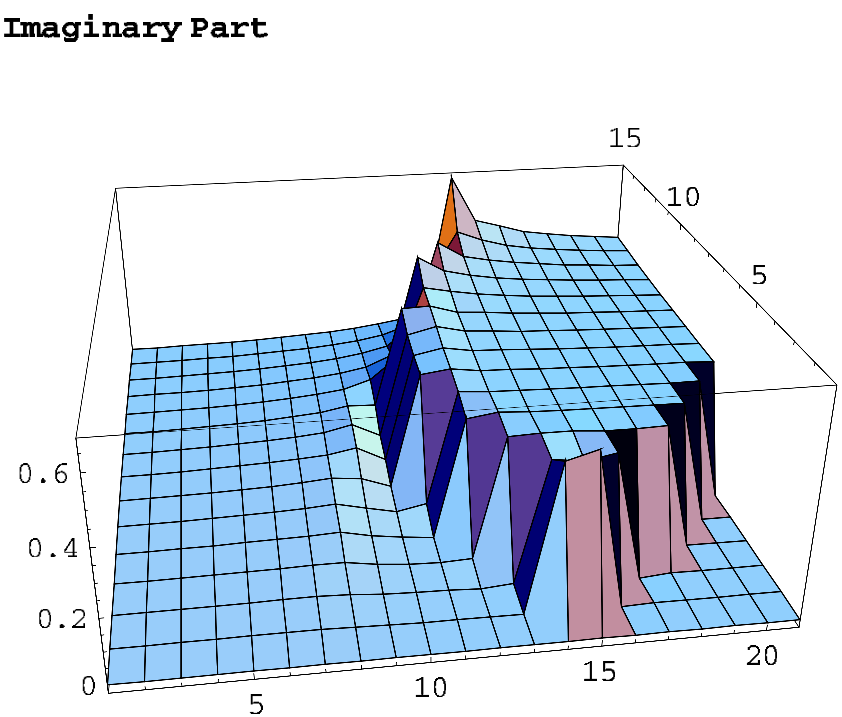


• Real part of off-shell amplitude



$Re[F_0(p_0, p; p'')]$ vs. p, p_0 at $p'' = 0.5$, $\alpha = 0.5$, $\mu = 0.5$.

Imaginary part of off-shell amplitude



$Im[F(p_0, p; p'')]$ vs. p, p_0 at $p'' = 0.5$, $\alpha = 0.5$, $\mu = 0.5$.

• Conclusions

- The Bethe-Salpeter amplitude in Minkowski space, for the bound states, both for spinless particles and fermions, is found.
- E.M. form factors are analytically expressed through a computed non-singular Nakanishi functions $g_i(\gamma, z)$.
- The Bethe-Salpeter off-shell amplitude in Minkowski space, for the scattering states, both for spinless particles, is also found (by a different method).

• What should be done

- To apply the off-shell BS scattering amplitude to the calculating the transition form factors.
- To extend the "direct treatment of singularities" method to fermions (no new difficulties foreseen, a technical task).
- To calculate the off-shell BS scattering amplitude via Nakanishi representation (both for spineless particles and fermions). Advantages:
  - No need in the partial wave decomposition.
  - Calculation of singular integrals in form factors is analytical.
- Analyze the realistic few-nucleon systems in the BS framework (bound states, phase shifts, off-shell amplitudes, e.m. form factors, nuclear reactions,.)
- To develop BS-Faddeev three-body approach.

Thank you!