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Nonperturbative calculations in truncated Fock space in LFD

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• Outline

- Fock space, its truncation and eigenvector equation.
- Sector-dependent renormalization.
- Illustration: "0D field theory".
- Yukawa Model. E.M. form factors and anomalous magnetic moment.

• Eigenstate equation:

$$H\,|p\rangle = M\,|p\rangle$$

In field theory Hamiltonian H does not conserve the number of particles.
Solution $|p\rangle$ of the eigenstate equation is superposition of the states with different numbers of particles.

Hamiltonian H and the state vector $|p\rangle$ are defined the the LFD framework.

• State vector

The state vector is represented as the (exact) Fock decomposition:

$$|p\rangle = \sum_{n=1}^{\infty} \int \psi_n(k_1, \ldots, k_n, p)\, |n\rangle\, D_k$$

It contains infinite number of the Fock components ψ_n.
Approximation: replace this sum by the finite one (truncation):

$$|p\rangle = \sum_{n=1}^{N} \int \psi_n(k_1, \ldots, k_n, p)\, |n\rangle\, D_k$$

An alternative to the lattice calculations?

• Eigenvalue equation:

$$H\,|p\rangle = M\,|p\rangle$$

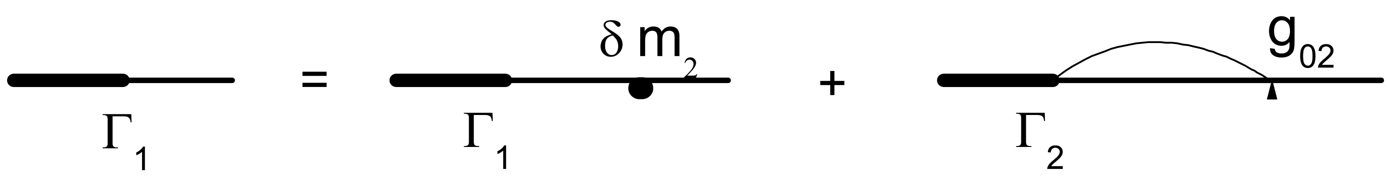
It results in a system of equations for the Fock components ψ_n.

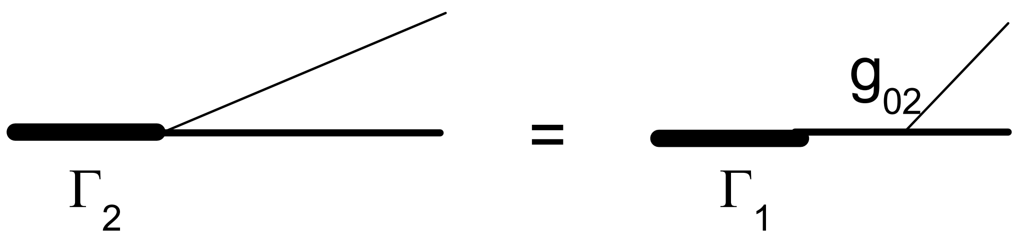
$$\begin{pmatrix} H_{11} & H_{12} & 0 & 0 \\ H_{21} & H_{22} & \dots & 0 \\ 0 & \dots & \dots & \dots \\ 0 & 0 & \dots & H_{NN} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_N \end{pmatrix} = M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_N \end{pmatrix}$$

The coupling constant α in H_{ij} may be large. After truncation, the numerical solution of the system of equations is non-perturbative.

The solution requires renormalization.

• Two-body truncation

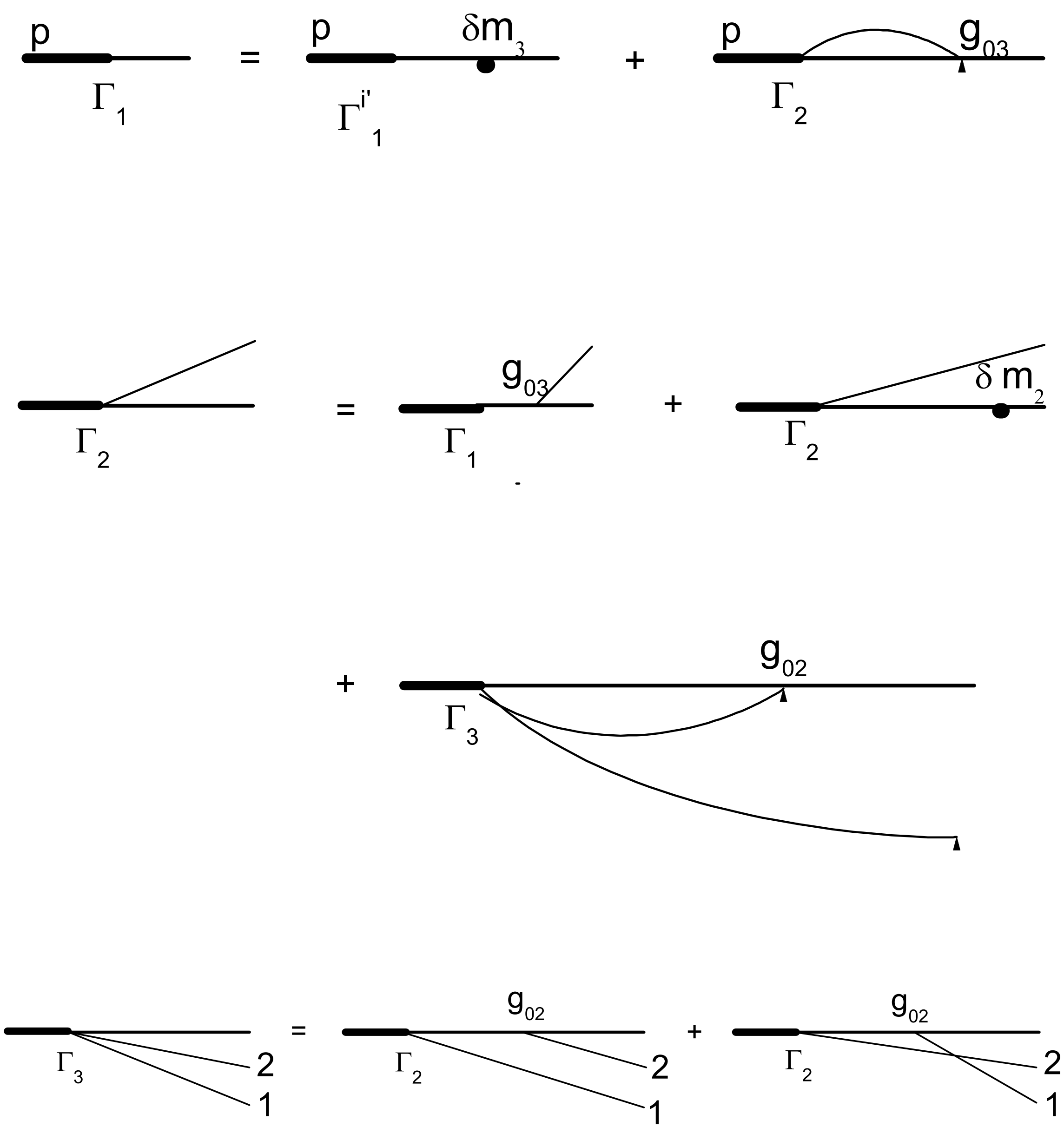




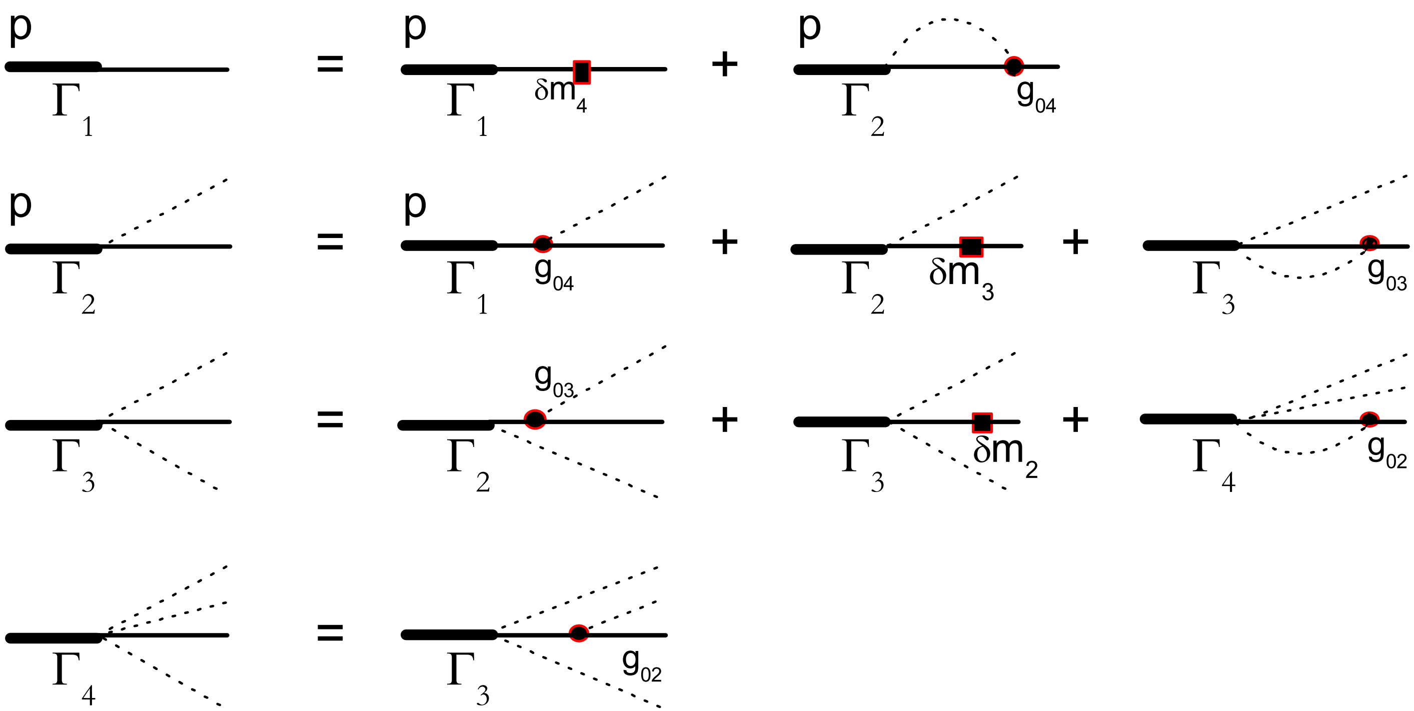
System of equations for physical and Pauli-Villars particles
(one PV fermion and one PV boson).

• Three-body truncation

System of equations

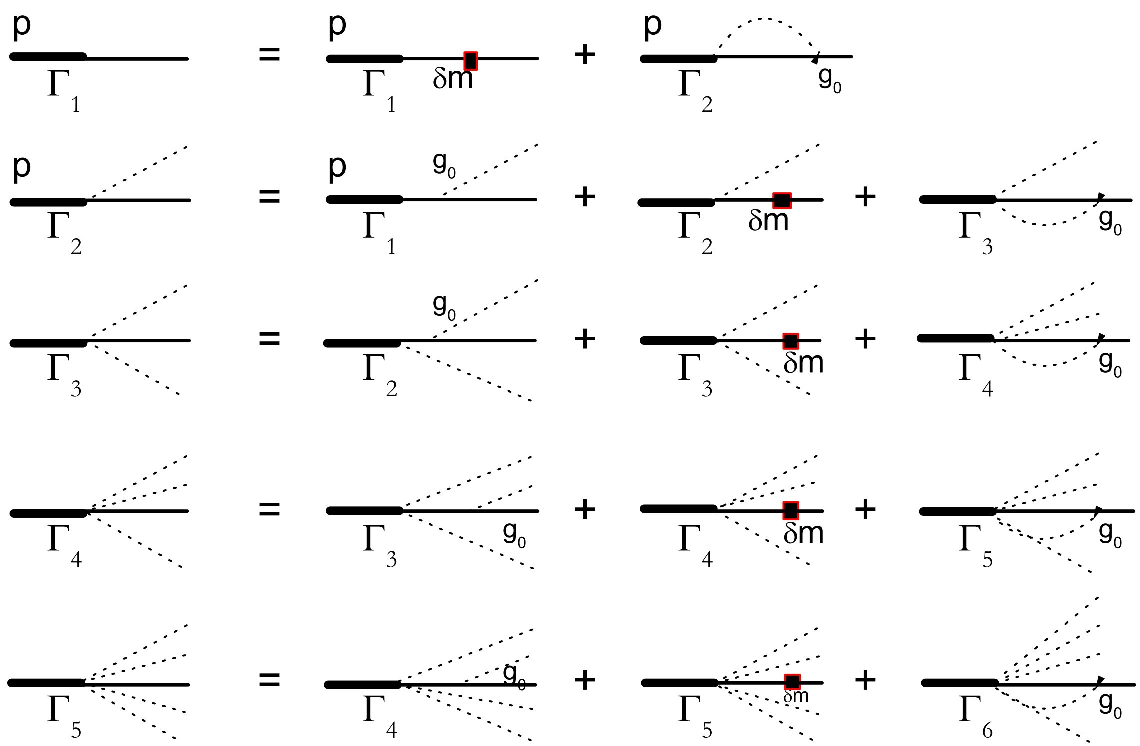


• Higher Fock sectors ($N = 4$)



System of equations for the vertex functions Γ_{1-4}.

• Higher Fock sectors ($N = 6$)



System of equations for the vertex functions Γ_{1-6}.

Equations always couple the near by components: Γ_n with $\Gamma_{n\pm1}$.

• Two principal problems

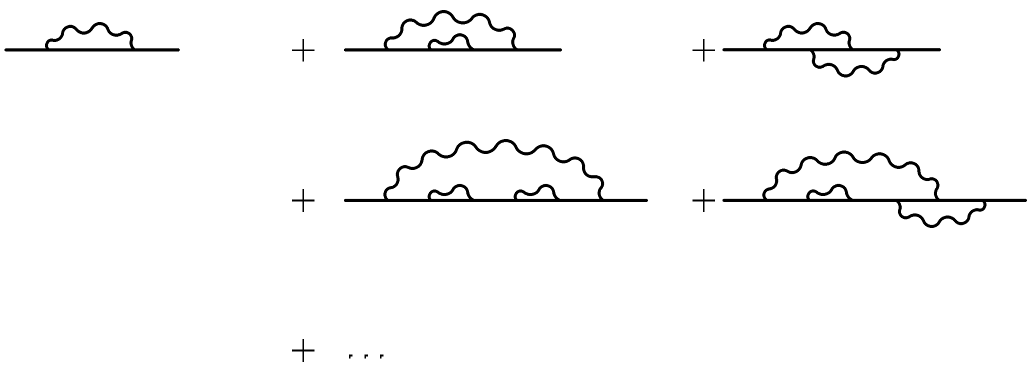
1. Quick increase of the matrix dimension.
   Solution - supercomputers.
2. Infinities.
   Solution - sector dependent renormalizaton.

• Renormalization

Parameters in Hamiltonian: bare coupling constant g_0 and mass counter terms δm must be expressed (from the renormalization conditions) through physical (observed) coupling constant g_0 and physical mass m.

As a by-product, in any given order of perturbation theory, infinities canceled.

• Example: Three-body self-energy



Perturbative expansion, in terms of the pion-nucleon coupling constant g, of the nucleon self-energy.

• Consequences

- All the states with $N \leq 3$.
- For given order of g – not full set of perturbative graphs (since some graphs exceed $N = 3$).

Therefore, infinities, after renormalization,
are not cancelled.

• Sector-dependent renormalization

To provide cancellations of infinities

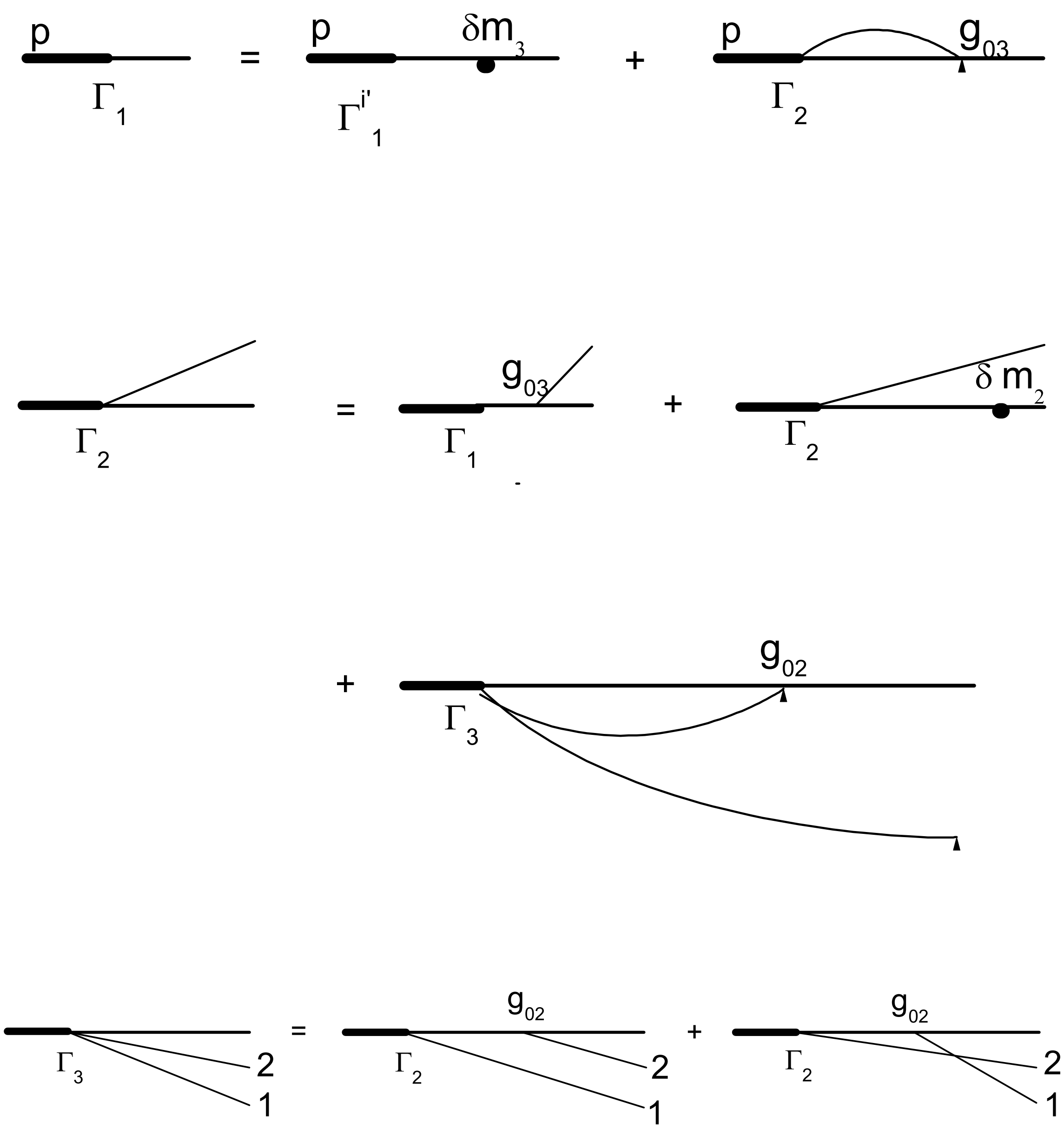
R. Perry, A. Harindranath, K. Wilson,
Phys. Rev. Lett. 24 (1990) 2959.

V.A. Karmanov, J.-F. Mathiot, A.V. Smirnov,
Phys. Rev. **D77** (2008) 085028.

Main aim of this talk is to illustrate
the sector-dependent renormalization
on example of zero-dimensional "field theory".

• Three-body truncation

System of equations



• 0D "field theory"

General form of equation:

$$\begin{pmatrix} \delta\, m & V_{12} & 0 & 0 & \ldots \\ V_{21} & \delta\, m-1 & V_{23} & 0 & \ldots \\ 0 & V_{32} & \delta m-1 & V_{34} & \ldots \\ 0 & 0 & V_{43} & \delta\, m-1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \ldots \end{pmatrix} = 0$$

Replace the integral terms by the products: $\int V_{ij}\psi_j \ldots \to V_{ij}\psi_j$.

• Interaction V_{ij}

$$V_{ij} = \frac{g}{(i+j)}$$ – higher sectors suppressed

$N = 2$ truncation

No any sector dependent scheme., $g = 2$.

$$\begin{pmatrix} \delta m & V_{12} \\ V_{21} & \delta m - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} \delta m & \frac{2}{3} \\ \frac{2}{3} & \delta m - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

$$det \begin{pmatrix} \delta m & \frac{2}{3} \\ \frac{2}{3} & \delta m - 1 \end{pmatrix} = 0 \quad \Rightarrow \quad \delta m^2 - \delta m - \frac{4}{9} = 0$$

Quadratic equation, two solutions:

$$\delta m = \frac{1}{2} \mp \frac{5}{6} \quad \rightarrow \quad \delta m = -0.333, \quad \delta m = 1.333$$

• $N = 3$ truncation

No any sector dependent scheme.

$$\begin{pmatrix} \delta m & \frac{2}{3} & 0 \\ \frac{2}{3} & \delta m - 1 & \frac{2}{5} \\ 0 & \frac{2}{5} & \delta m - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0$$

Cubic equation: $\delta m^3 - 2\delta m^2 + \dfrac{89}{225}\delta m + \dfrac{4}{9} = 0$

Three solultions:

$$\begin{aligned} \delta\, m &= -0.358 \\ \delta\, m &= 0.792 \\ \delta\, m &= 1.566 \end{aligned}$$

Reminder: $N = 2$ truncation: $\delta m = -0.333, \quad \delta m = 1.333.$

• Truncations $N = 1, 2, \ldots 10$

N	δm
1	$0.$
2	-0.333
3	-0.3582
4	-0.35948
5	-0.3595154
6	-0.3595160702
7	-0.35951607879
8	-0.3595160788796
9	$-0.359516078880294 1$
10	-0.3595160788802980

Solutions were found solving (non-linear) polynomial equation of the Nth degree.
Solutions for $N = 10$ were found solving polynomial equation of the 10th degree.

When $N \to \infty$, it approaches to "exact" solution.
Exact solution is given by the $N = 10$ solution with precision in 14 digits.

• State vector

$N = 8$ truncation

$$\psi = \begin{pmatrix} 0.870 \\ 0.469 \\ 0.145 \\ 0.312 \cdot 10^{-1} \\ 0.520 \cdot 10^{-2} \\ 0.705 \cdot 10^{-3} \\ 0.806 \cdot 10^{-4} \\ 0.790 \cdot 10^{-5} \end{pmatrix}$$

Normalization: $\langle \psi | \psi \rangle = \psi_1^2 + \psi_2^2 + \ldots + \psi_8^2 = 1.$

In red – components which exceed 0.1.

We see that higher sectors decrease.

• Large coupling constant $g = 8$

n	δm_n
1	0
2	-7.1111
3	-1.2810
4	-3.7331
5	-2.0226
6	-2.9588
7	-2.3553
8	-2.7101
9	-2.4889
10	-2.6221
11	-2.5401
12	-2.5899
13	-2.5595
14	-2.5780
exact	-2.57097

Converges more slowly.

• Corresponding state vector ($g = 8$)

$$\psi = \begin{pmatrix} 0.675 \\ 0.651 \\ 0.327 \\ 0.112 \\ 0.291 \cdot 10^{-1} \\ 0.612 \cdot 10^{-2} \\ 0.108 \cdot 10^{-2} \\ 0.161 \cdot 10^{-3} \end{pmatrix}$$

More first components dominate.

• Quickly decreasing kernel

No any sector dependent scheme.
Direct solution with $V_{ij} = \frac{2^4}{(i+j)^4}$

N	$\delta\, m$
1	0
2	-0.03760
3	-0.037626467
4	-0.03762645766827
5	$-0.0376264576682328628 84$
6	$-0.037626457668232862889731262 8$
7	$-0.03762645766823286288973126446805915$
8	$-0.0376264576682328628897312644680593003 85$
9	$-0.0376264576682328628897312644680593003 85$
10	$-0.0376264576682328628897312644680593003 85$

40-digits stability is achieved at $N = 8$ truncation.

Superfast convergence!

• State vector

No any sector dependent scheme.
Direct solution with $V_{ij} = \frac{2^4}{(i+j)^4}$

$$\psi = \begin{pmatrix} 0.982 \\ 0.187 \\ 0.462 \cdot 10^{-2} \\ 0.296 \cdot 10^{-4} \\ 0.696 \cdot 10^{-7} \\ 0.734 \cdot 10^{-10} \\ 0.396 \cdot 10^{-13} \\ 0.127 \cdot 10^{-16} \end{pmatrix}$$

One- and two-body components dominate.

$N = 2$ truncation (sector dependent)

$$\begin{pmatrix} \delta m & \frac{2}{3} \\ \frac{2}{3} & \delta m - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} \delta m_2 & \frac{2}{3} \\ \frac{2}{3} & \delta m_1 - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

$$det \begin{pmatrix} \delta m & \frac{2}{3} \\ \frac{2}{3} & \delta m - 1 \end{pmatrix} = 0 \quad \Rightarrow \quad det \begin{pmatrix} \delta m_2 & \frac{2}{3} \\ \frac{2}{3} & \delta m_1 - 1 \end{pmatrix} = 0$$

$$\delta m_1 = 0 \quad \Rightarrow \quad det \begin{pmatrix} \delta m_2 & \frac{2}{3} \\ \frac{2}{3} & -1 \end{pmatrix} = 0$$

Linear equation $\delta m_2 + \left(\frac{2}{3}\right)^2 = 0$, one solution.

$$\delta m = -0.333, \quad 1.333 \quad \Rightarrow \quad \delta m_2 = -\frac{4}{9} = -0.444$$

$N = 3$ truncation (sector dependent)

$$\begin{pmatrix} \delta m_3 & \frac{2}{3} & 0 \\ \frac{g}{3} & \delta m_2 & \frac{2}{5} \\ 0 & \frac{2}{5} & \delta m_1 - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0$$

Since are known : $\delta m_1 = 0, \delta m_2 = -4/9$,
δm_3 is found from the linear equation:

$$det \begin{pmatrix} \delta m_3 & \frac{2}{3} & 0 \\ \frac{2}{3} & \delta m_2 & \frac{2}{5} \\ 0 & \frac{2}{5} & \delta m_1 - 1 \end{pmatrix} = 0 \quad \rightarrow \quad \delta m_3 \left(\frac{13}{9} - \frac{4}{25} \right) + \frac{4}{9} = 0$$

Linear equation for δm_3:

$$g = 2 \quad \rightarrow \quad \frac{289}{225} \delta m_3 + \frac{4}{9} = 0 \quad \rightarrow \quad \delta m_3 = -\frac{100}{289} = -0.346$$

Etc.

$N = 8$ truncation (sector dependent)

n	$\delta\, m_n$
1	0
2	-0.444
3	-0.346
4	-0.3617
5	-0.35915
6	-0.359574
7	-0.359506
8	-0.359517
exact	-0.3595161

All of them are found from linear equation.

This allows to cancel infinities

• Renormalization → subtraction

- Sector-dependent scheme converges to the exact one.
- For any N the counter terms is found from linear equation!

In perturbation theory the infinities are canceled due to subtractions of counter terms. Example: self energy.

$$\Sigma(\not{k}) \to \Sigma_{ren}(\not{k}) = \Sigma(\not{k}) - \Sigma(\not{k} = m) - (\not{k} - m) \left. \frac{\partial \Sigma(\not{k})}{\partial \not{k}} \right|_{\not{k}=m}$$

It is difficult to imagine any other mechanism of cancellation of divergences in the integrals.

• Straightforward way

Following straightforward way, we find the counter term as a non-linear function of divergent integral (like roots of polynomial of 10th degree in the above examples).

Substitute it back in equation:

$$\ldots \delta m(\text{nonlinear function of a divergent integral}) \ldots$$
$$- \int (\text{divergent integral}) = 0$$

How can they be canceled?

• Sector-dependent scheme

Following sector-dependent renormalization scheme, in each sector we find the counter term as solution of linear equation!
Being substituted back in the system of equations for the Fock components, they can provide the cancellation of infinities.

This is just what we observe in Yukawa model!

That's why we use the sector-dependent renormalization scheme.

• Explicitly covariant LFD

V.A. Karmanov, JETP, **44** (1976) 201.

J. Carbonell, B. Desplanques, V.A. Karmanov, J.-F. Mathiot, Phys. Reports, **300** (1998) 215.

$$t+z=0 \quad \rightarrow \quad \omega{\cdot}x=\omega_0 t-\vec{\omega}{\cdot}\vec{x}$$

where $\omega=(\omega_0,\vec{\omega})$ such that $\omega^2=0$.

The unit vector $\vec{n}=\frac{\vec{\omega}}{|\vec{\omega}|}$ determines the orientation of the light-front plane.

Particular case: $\omega=(1,0,0,-1)$ corresponds to the standard approach.

• Renormalization condition

Reminder: $\bar{u}(k)\Gamma_2 u(p) = \bar{u}(k)\left[b_1 + \frac{m\not\omega}{\omega\cdot p} b_2\right] u(p)$

On energy shell $s = m^2$ we should impose:

1. $b_1(s = m^2) = g$ (relation between g_{03} and g)
2. $b_2(s = m^2) = 0$ (kills ω-dependence in Γ_2).
3. $M = m_{phys}$ (determines δm_3.)

To satisfy 2., we introduce the ω-dependent counter term by

$$g_{03} \to g_{03} + \frac{m\ \not\omega}{\omega\cdot p} Z_\omega$$

• Determining counter terms

We must find

not only δm_n, but also g_{0n} and $Z_{\omega,n}$.

• x-dependent counter terms

However: $s = (k_1 + k_2)^2 = \frac{k_\perp^2 + \mu^2}{x} + \frac{k_\perp^2 + m^2}{1-x} = m^2$

This means $k_\perp^2 = -x^2 m^2 - (1-x)\mu^2 < 0$

(non-physical x -dependent value)

Renormalization condition:

$$b_1^{i=0,j=0}(g_{03}; k_\perp(x), x) = g, \quad k_\perp(x) = i\sqrt{x^2 m^2 - (1-x)\mu^2}$$

$b_1^{i=0,j=0}(g_{03}; k_\perp(x), x)$ depends on x because of truncation.

The same for the ω-dependent counter term: $Z_\omega = Z_\omega(x)$
to make $b_2^{i=0,j=0}(k_\perp, x) = 0$ at $s = m^2$, for any x.

Sector and x-dependent counter terms

St. Glazek, A. Harindranath, S. Pinsky, J. Shigemitsu, and K. Wilson, Phys. Rev. **D 47**, 1599 (1993).

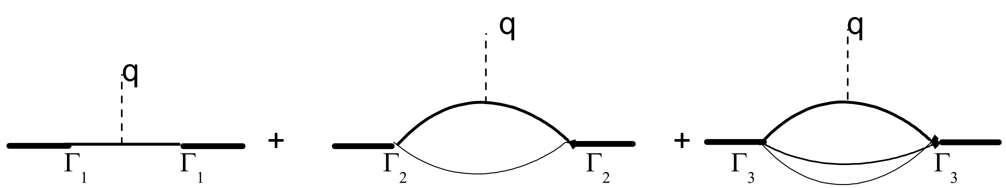
In the initial Hamiltonian, the counter terms do not depend on the Fock sectors and kinematical variables.
Making truncation, we replace the initial Hamiltonian by a finite matrix.

The counter terms naturally depend on the dimension of matrix (sector dependence) and on kinematical variables (x-dependence).

Inspite of that, the counter terms are found absolutely unambiguously.
They (hopefully) provide finite results after non-perturbative renormalization.

• EM form factors

1- and 2-body components are found from equations (model-dependent).
3-body components are expressed through 2-body components (model-dependent).
Form-factors are expressed through 1-, 2- and 3-body components (model-independent).



1-, 2- and 3-body contributions in EM form factors

• Form factor F_1

V. A. KARMANOV, J.-F. MATHIOT, AND A. V. SMIRNOV PHYSICAL REVIEW D **86,** 085006 (2012)

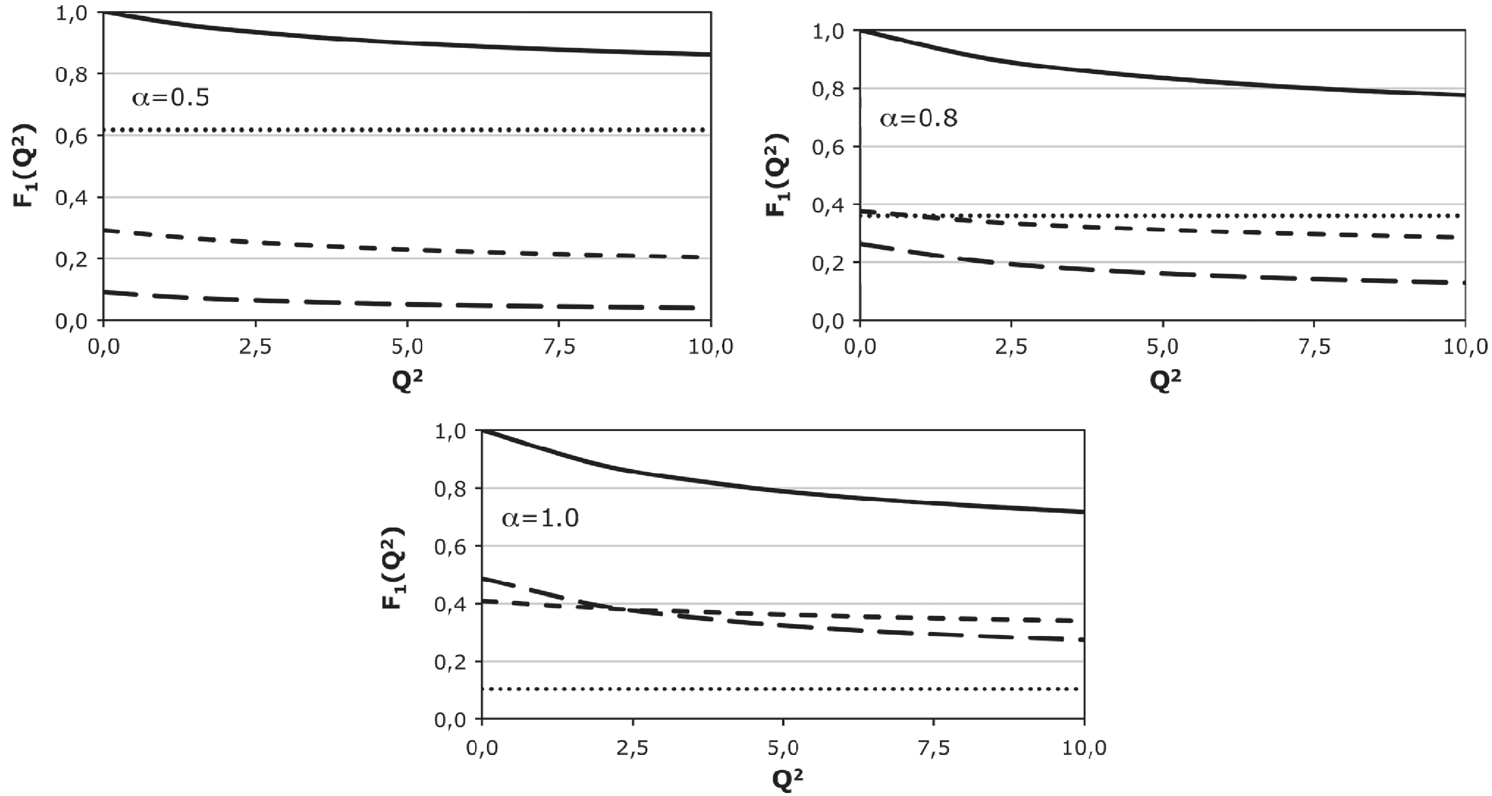


FIG. 7. Electromagnetic form factor $F_1(Q^2)$ in the Yukawa model, at $\mu_1 = 100$, for $\alpha = 0.5$ (upper left plot), 0.8 (upper right plot), and $\alpha = 1.0$ (lower plot). The dotted, dashed, and long-dashed lines are, respectively, the one-, two-, and three-body contributions, while the solid line is the total result.

Since h_i^j, H_i^j start growing from the characteristic values

The existence of a critical value for the regularization

• Anomalous magnetic moment

PHYSICAL REVIEW D **86,** 085006 (2012)

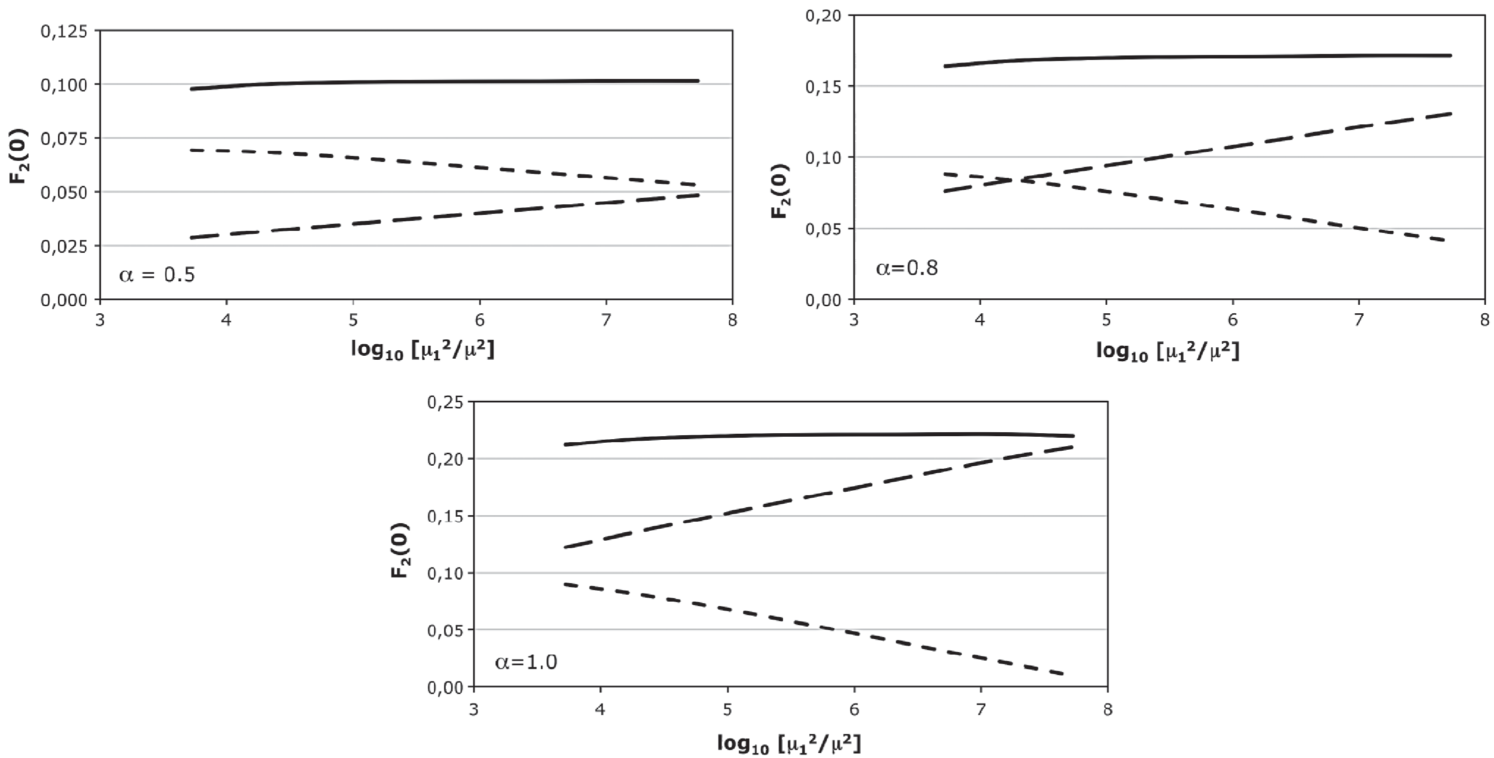


FIG. 6. The anomalous magnetic moment in the Yukawa model as a function of the PV mass μ_1, for three different values of the coupling constant, $\alpha = 0.5$ (upper left plot), 0.8 (upper right plot), and $\alpha = 1.0$ (lower plot). The dashed and long-dashed lines are, respectively, the two- and three-body contributions, while the solid line is the total result.

C. Numerical results

We finally show in Fig. 9 the contributions of the one-, two-, and three-body sectors to the norm of the state

• Adding antifermion ($f\,f\,\bar{f}$)

x-dependent counter term $Z_\omega(x)$

Dashed line – without $f\,f\,\bar{f}$. Solid line – with $f\,f\,\bar{f}$.

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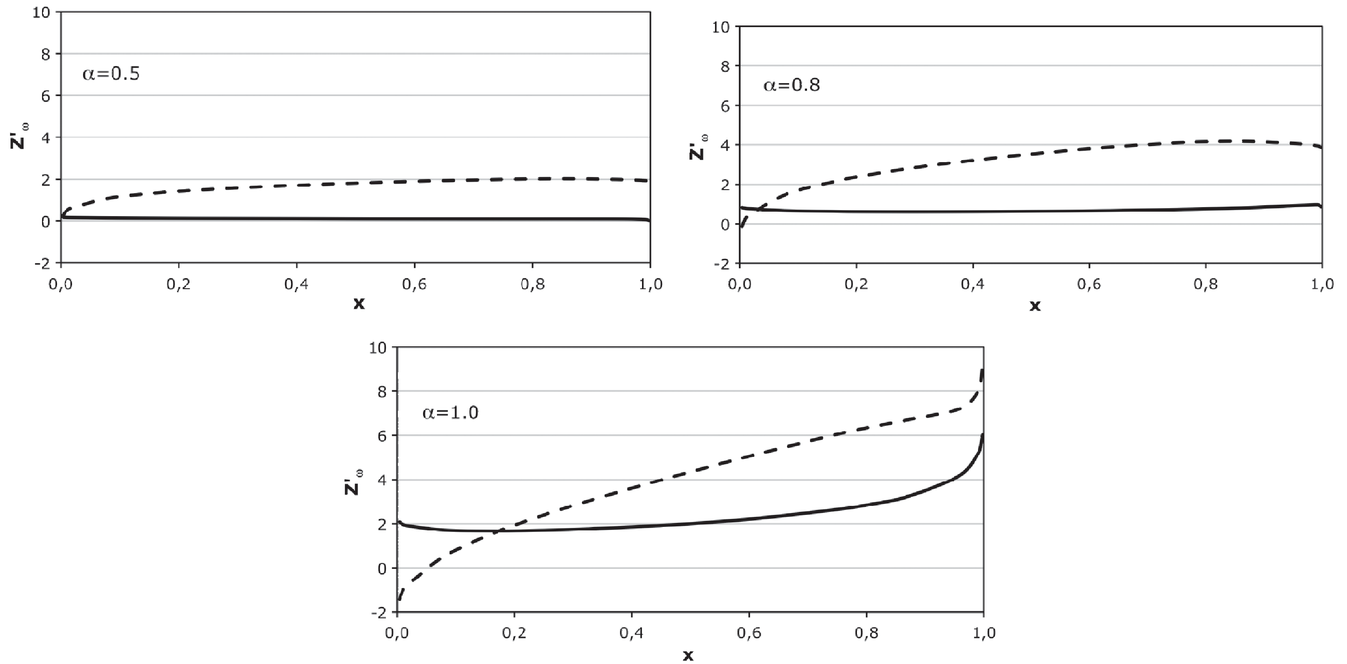
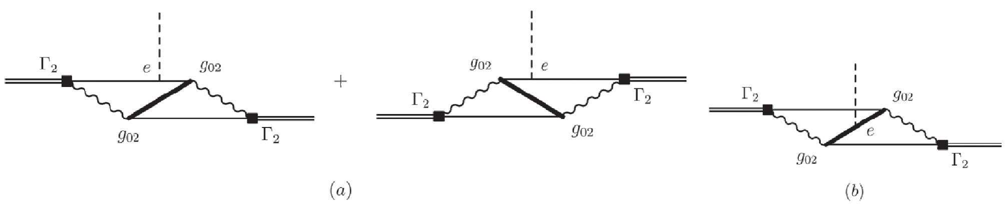
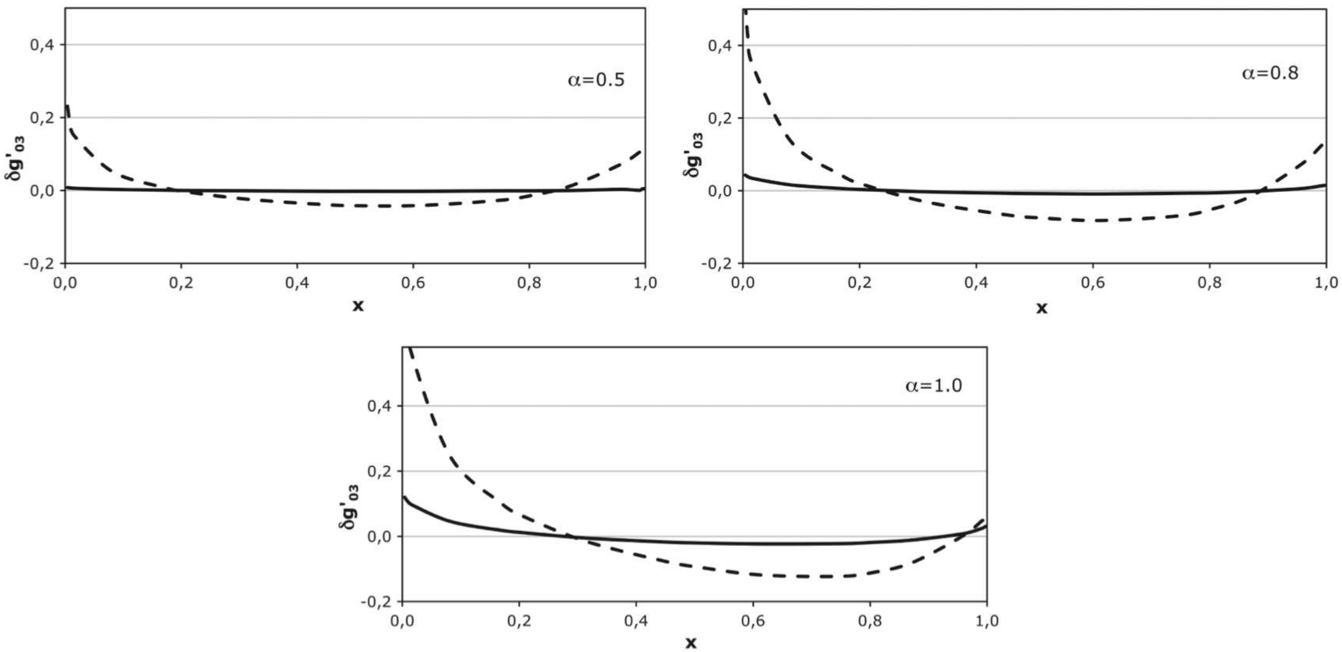


FIG. 14. x dependence of the counterterm Z'_ω for $\alpha = 0.5$ (upper left plot), $\alpha = 0.8$ (upper right plot), and $\alpha = 1.0$ (lower plot), calculated for $\mu_1 = 100$. The solid (dashed) lines correspond to the results obtained with (without) the $ff\bar{f}$ Fock sector contribution.



• Bare coupling constant $g_{03}(x)$

$$\delta g_{03}(x) = (g_{03}(x) - \bar{g}_{03})/\bar{g}_{03}$$



IG. 13. x dependence of the bare coupling constant g'_{03}, calculated relatively to its mean value over the interval $x \in [0, 1]$, for $= 0.5$ (upper left plot), $\alpha = 0.8$ (upper right plot), and $\alpha = 1.0$ (lower plot), calculated for $\mu_1 = 100$. The solid (dashed) lines orrespond to the results obtained with (without) the $ff\bar{f}$ Fock sector contribution.

Dashed line – without $ff\bar{f}$. Solid line – with $ff\bar{f}$.

• Everything goes in good direction!

- Form factors do not depend on the PV masses when the latter tend to infinity – convergence.
- x-dependent counter terms become flat – stop to depend on x – when we increase the number of truncated states.

• Conclusion

- Non-perturbative approach, based on the truncation of Fock space, is developed.
- Fock space is truncated up to three-body states, including state with antifermion ($ff\bar{f}$).
- E.M. form factors and anomalous magnetic moment are calculated in the Yukawa model.
- The results are stable (i.e., they converge) vs. increase of the meson PV mass.
- It is the time go to higher truncations (by means of supercomputers).