# Light-front holography and the light-front coupled-cluster method

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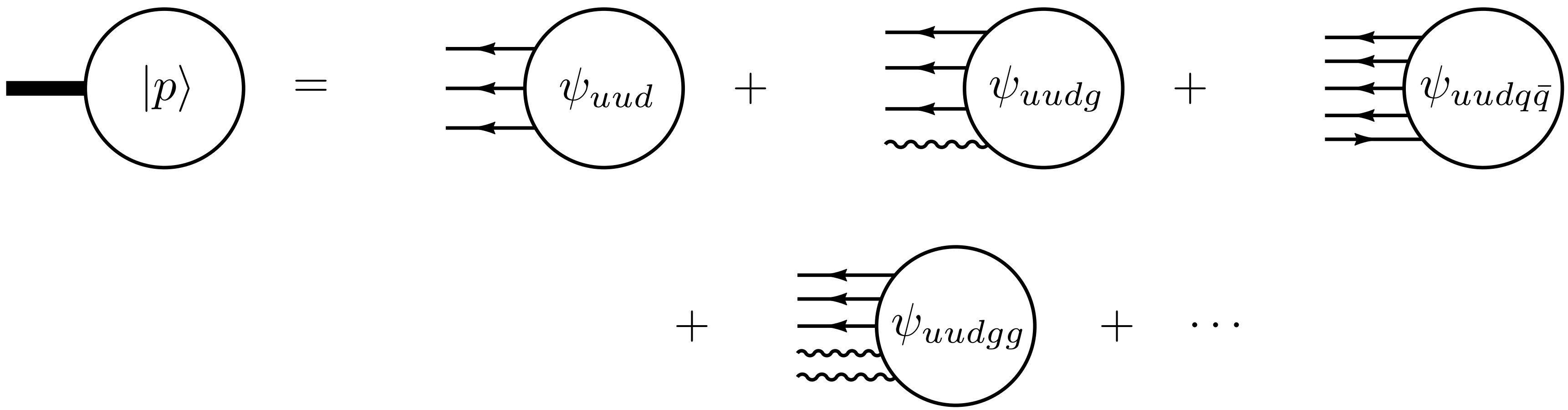
# Outline

- introduction
  - bound-state eigenvalue problem
- light-front coupled-cluster method
  - Chabysheva and jrh, PLB **711**, 417 (2012) arXiv:1203.0250, 1208.6503, 1208.6076.
    - also talk by Chabysheva.
  - sample application
  - form-factor calculation
- light-front holography
  - Brodsky and De Téramond, PRL **96**, 201601 (2006).
- quark model for mesons
  - Chabysheva and jrh, arXiv:1207.7128.
- summary

# Introduction

- wish to compute hadron structure in terms of wave functions

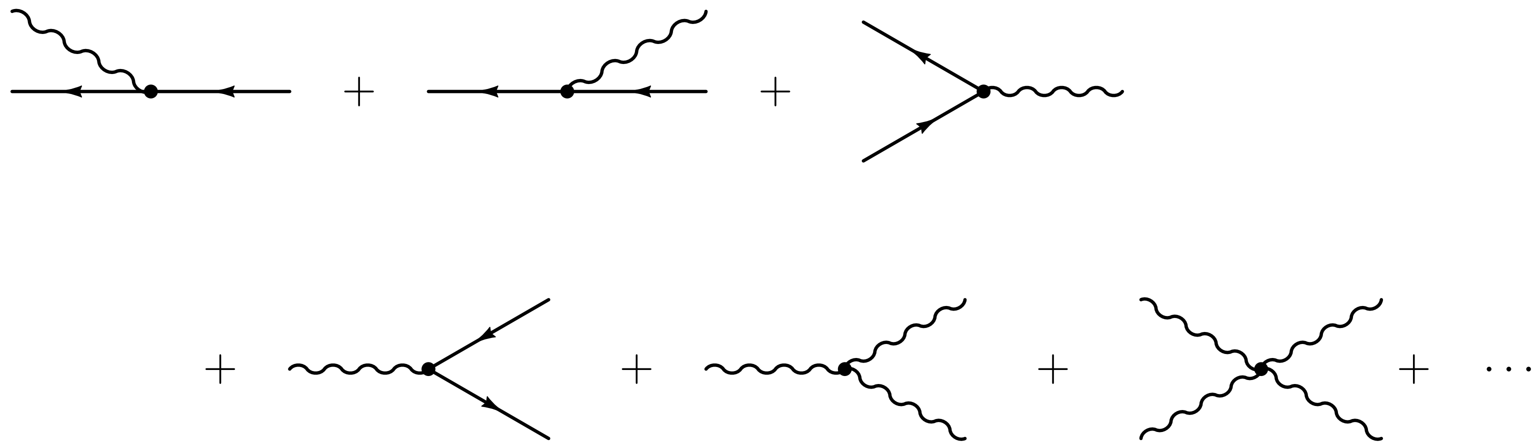



- must truncate in some fashion
- the LFCC method avoids Fock-space truncation

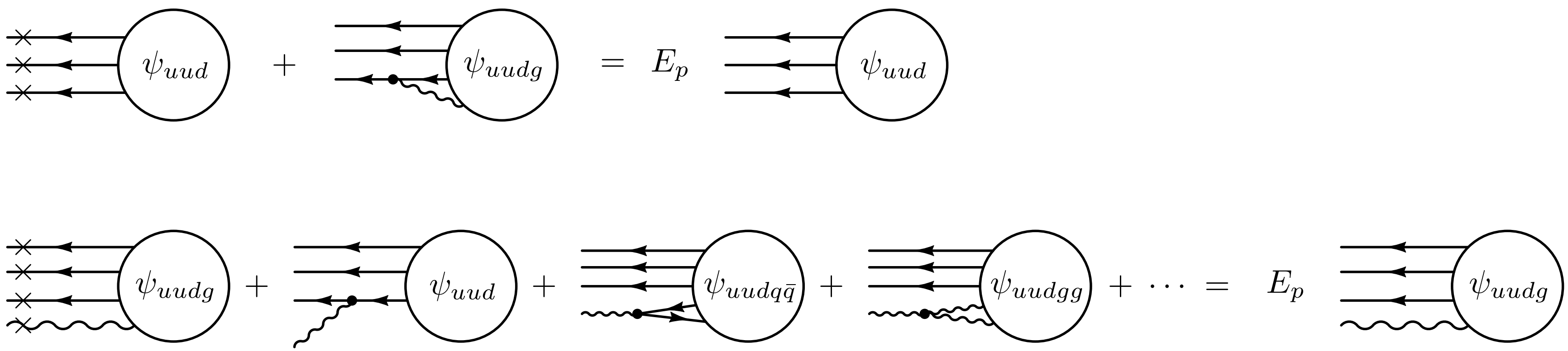
# Equations for wave functions

Solve $\left(\text{K.E.} + V_{\text{QCD}}\right)|p\rangle = E_p|p\rangle$ with $E_p = \sqrt{m_p^2 + p^2}$ and

$V_{\text{QCD}} =$



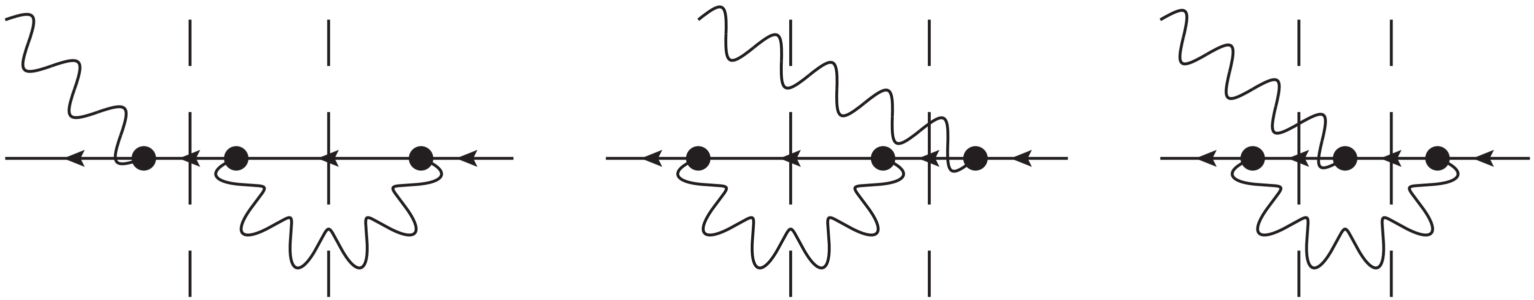
Equivalent to coupled integral equations




$\vdots$

# Uncanceled divergences

- for example, the Ward identity of gauge theories is destroyed by truncation

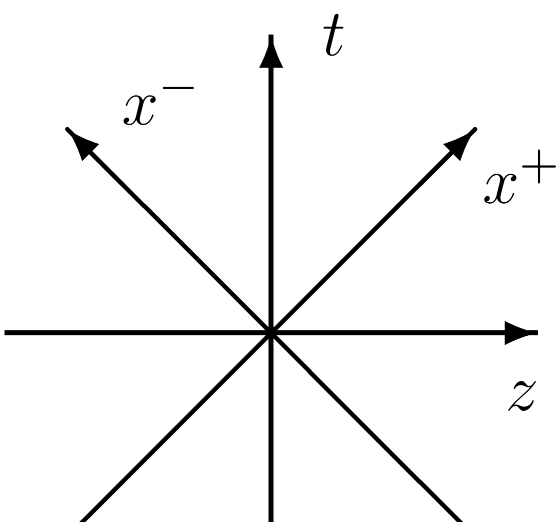


- analog in Feynman perturbation theory
  - separate diagrams into time-ordered diagrams
  - discard time orderings that include intermediate states with more particles than some finite limit
  - destroys covariance, disrupts regularization, and induces spectator dependence for subdiagrams
- in the nonperturbative case, this happens not just to some finite order in the coupling but to all orders

# Light-cone coordinates

Dirac, RMP **21**, 392 (1949);
Brodsky, Pauli, and Pinsky, Phys. Rep. **301**, 299 (1997).

- time: $x^+ = t + z$
- space: $\underline{x} = (x^-, \vec{x}_\perp), \quad x^- \equiv t - z, \quad \vec{x}_\perp = (x, y)$
- energy: $p^- = E - p_z$
- momentum: $\underline{p} = (p^+, \vec{p}_\perp), \quad p^+ \equiv E + p_z, \quad \vec{p}_\perp = (p_x, p_y)$
- mass-shell condition: $p^2 = m^2 \quad \Rightarrow \quad p^- = \frac{m^2 + p_\perp^2}{p^+}$



# Mass eigenvalue problem

- Pauli and Brodsky, PRD **32**, 1993 (1985); 2001 (1985)

$$\mathcal{P}^-|\underline{P}\rangle = \frac{M^2+P_\perp^2}{P^+}|\underline{P}\rangle, \quad \underline{\mathcal{P}}|\underline{P}\rangle = \underline{P}|\underline{P}\rangle.$$

- no spurious vacuum contributions to eigenstates
  - $p^+ > 0$ for all particles
  - cannot produce particles from vacuum and still conserve $p^+$
  - (but difficult to analyze structure of physical vacuum)
- boost-invariant separation of internal and external momenta
  - longitudinal momentum fractions $x_i \equiv p_i^+/P^+$
  - relative transverse momenta $\vec{k}_{i\perp} \equiv \vec{p}_{i\perp} - x_i\vec{P}_\perp$

# Coupled-cluster (CC) method

- originated with Coester, Nucl. Phys. **7**, 421 (1958) and Coester and Kümmel, Nucl. Phys. **17**, 477 (1960), with applications to the many-body Schrödinger equation in nuclear physics.
- applied to many-electron problem in molecules by Čižek, J. Chem. Phys. **45**, 4256 (1966).
- form eigenstate as $e^T|\phi\rangle$ where
  - $|\phi\rangle$ is product of single-particle states
  - terms in $T$ annihilate states in $|\phi\rangle$ and create excited states, to build in correlations
  - truncate $T$ at some number of excitations
- review: RJ Bartlett and M Musial, RMP **79**, 291 (2007).

# Light-front coupled-cluster method

- wish to solve $\mathcal{P}^-|\psi\rangle = \frac{M^2+P_\perp^2}{P^+}|\psi\rangle$.
- write eigenstate as $|\psi\rangle = \sqrt{Z}e^T|\phi\rangle$
  - $Z$ controls normalization: $\langle\psi'|\psi\rangle = \delta(\underline{P}' - \underline{P})$.
  - $|\phi\rangle$ is the valence state, with $\langle\phi'|\phi\rangle = \delta(\underline{P}' - \underline{P})$.
  - $T$ contains terms that only increase particle number.
  - $T$ conserves $J_z$, light-front momentum $\underline{P}$, charge, ...
  - $p^+ > 0 \Rightarrow T$ must include annihilation and powers of $T$ include contractions.
- construct $\overline{\mathcal{P}^-} = e^{-T}\mathcal{P}^-e^T$ and let $P_v$ project onto the valence Fock sector. Then, have coupled system:

$$P_v\overline{\mathcal{P}^-}|\phi\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi\rangle \quad \text{and} \quad (1-P_v)\overline{\mathcal{P}^-}|\phi\rangle = 0.$$

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# A soluble model

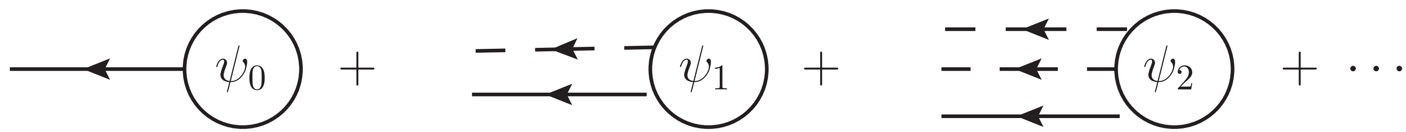
- light-front analog of the Greenberg–Schweber model
  - static fermionic source that emits and absorbs bosons without changing its spin

Chabysheva and jrh, PLB **711**, 417 (2012);
Brodsky, jrh, and McCartor, PRD **58**, 025005 (1998);
Greenberg and Schweber, N Cim **8**, 378 (1958).

- not fully covariant
  - hides some of the power of the LFCC method, but is sufficient to show how the method can be applied.
  - states are all limited to having a fixed total transverse momentum $\vec{P}_\perp$, which we take to be zero.

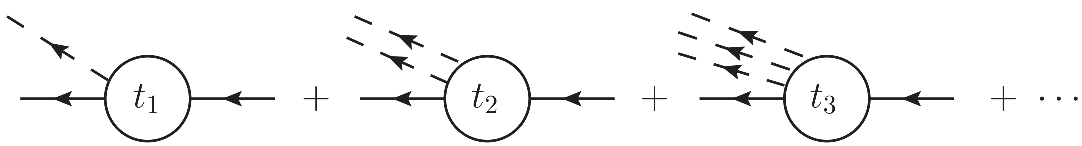
# Fock-state expansions

$|\psi\rangle =$

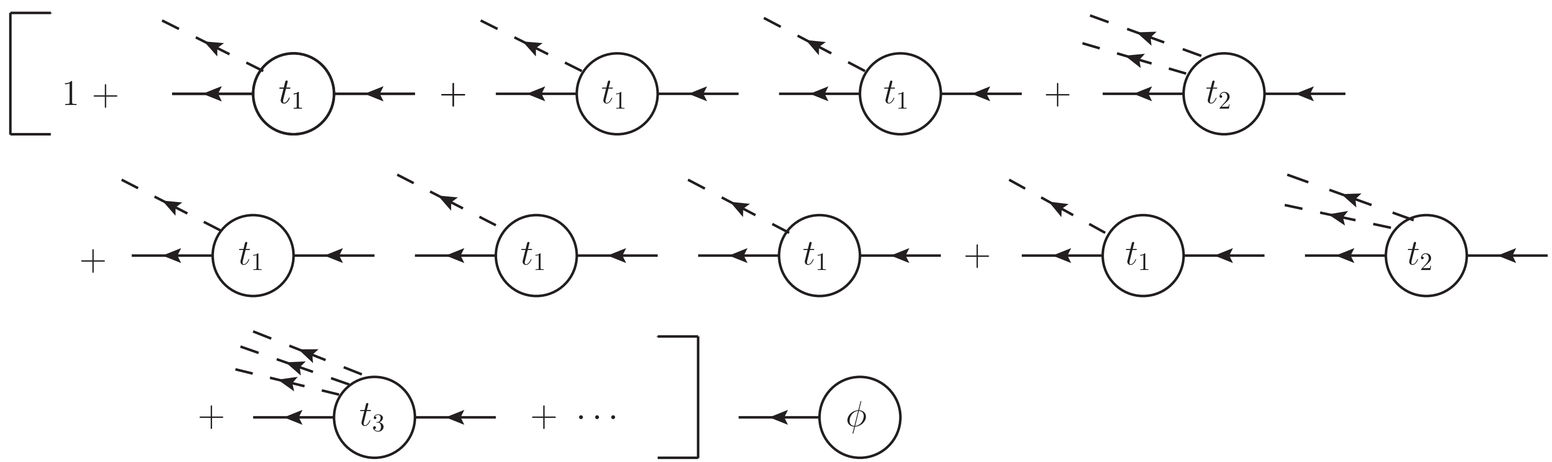



Instead, in LFCC:

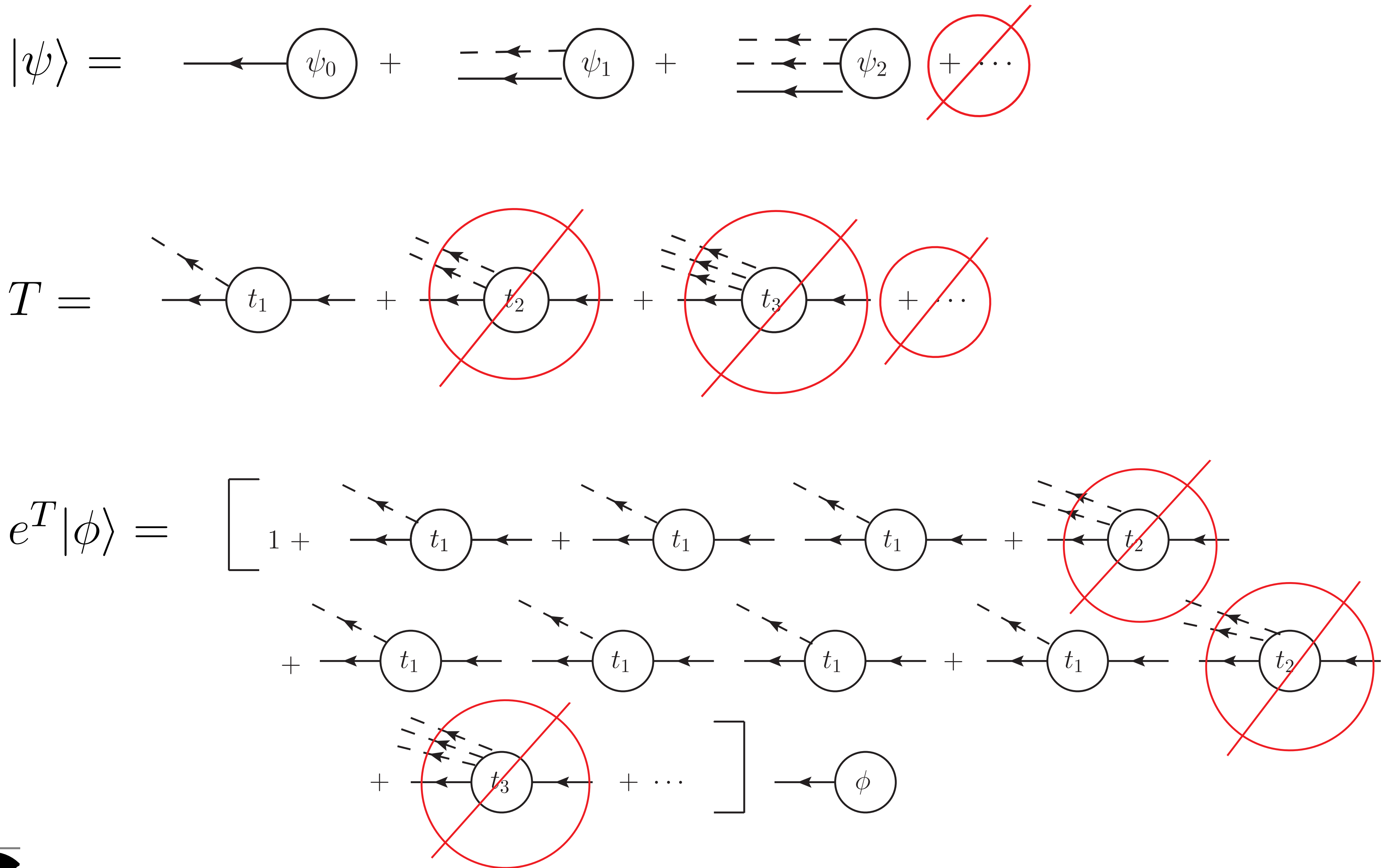
$T =$




$e^T|\phi\rangle =$



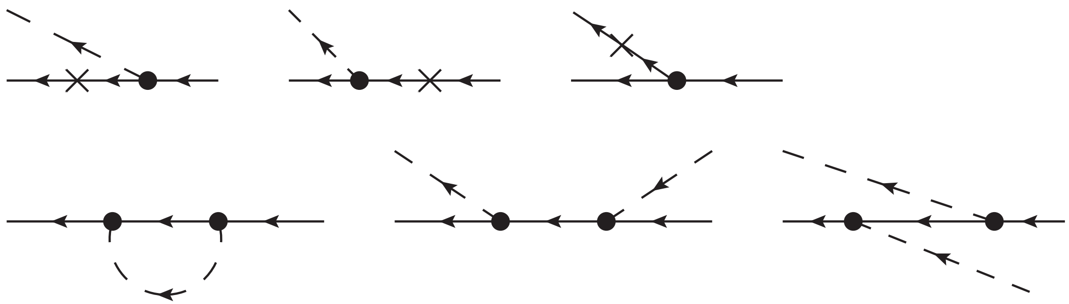
# Truncation



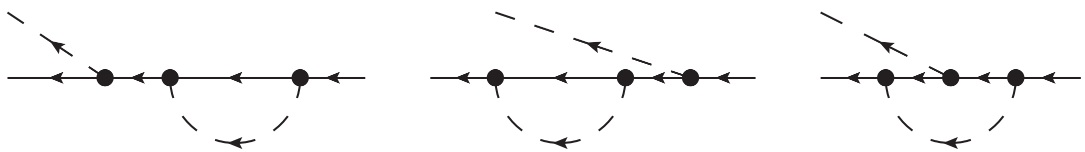
# Effective Hamiltonian

- constructed from Baker–Hausdorff expansion.

$$[\mathcal{P}^-, T] \to$$

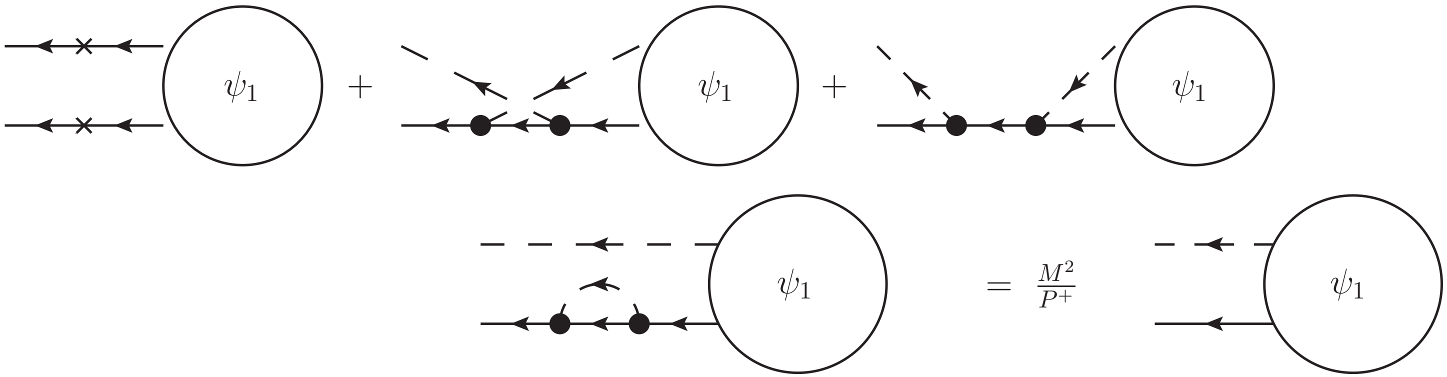


$$[[\mathcal{P}^-, T], T] \to$$

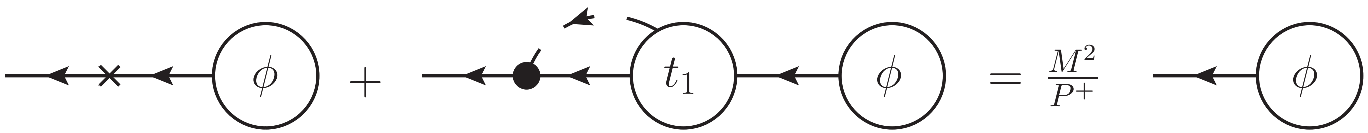


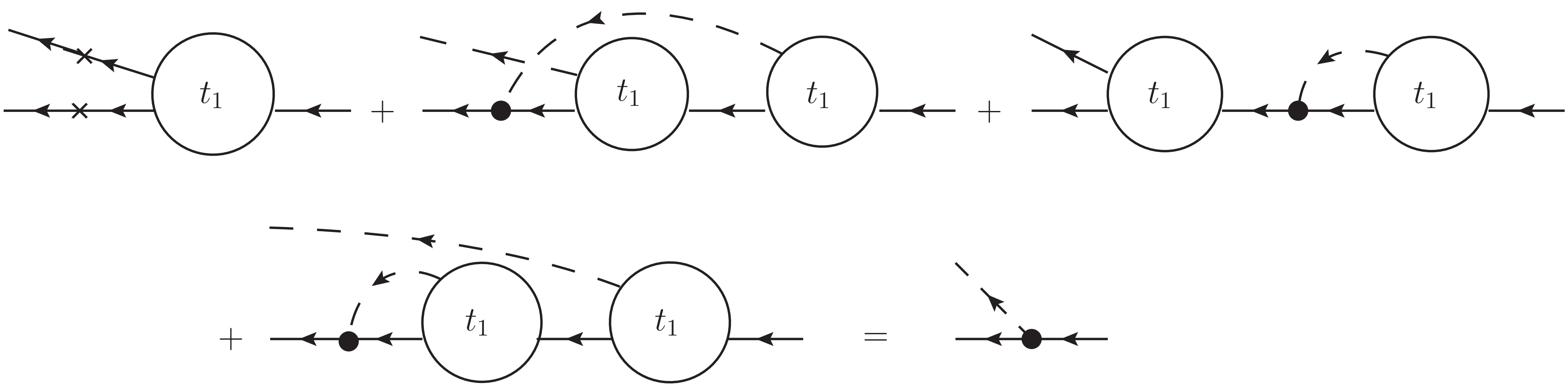
- list only terms that connect the lowest Fock sectors.
- the self-energy contribution is the same in all Fock sectors
- contains all three of the diagrams analogous to those for the Ward identity in QED

# Eigenvalue problems




In LFCC:



# Exact solution

- in this special case, the exponential operator $e^T$ generates the exact solution, with $t_{ls}(\underline{q},\underline{p}) = \frac{-g}{\sqrt{16\pi^3 q^+}} \left(\frac{p^+}{p^+ + q^+}\right)^\gamma \frac{q^+/P^+}{\mu_l^2 + q_\perp^2}$.
- the fact that the self-energy loop is the same in the valence sector and the one-fermion/one-boson sector plays a critical role. It contributes

$$-\frac{g}{P^+}\int \frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}\theta(P^+ - q^+)\left(\frac{P^+ - q^+}{P^+}\right)^\gamma \sum_l (-1)^l t_{l\pm}(\underline{q}, \underline{P} - \underline{q}),$$

  regulated by the PV ($l=1$) term.
- the expression for the loop obtained in the valence sector is exactly what is needed to obtain the necessary cancellations.

# Dirac form factor

- compute the Dirac form factor for the dressed fermion from a matrix element of the current $J^+ = \overline{\psi}\gamma^+\psi$
- the current couples to a photon of momentum $q$
- the matrix element is generally

$$\langle\psi^\sigma(\underline{P}+\underline{q})|16\pi^3 J^+(0)|\psi^\pm(\underline{P})\rangle = 2\delta_{\sigma\pm}F_1(q^2) \pm \frac{q^1 \pm iq^2}{M}\delta_{\sigma\mp}F_2(q^2)$$

  with $F_1$ and $F_2$ the Dirac and Pauli form factors.
- in the present model, the fermion cannot flip its spin; therefore, $F_2$ is zero, and we investigate only $F_1$

# Expectation values

- expectation value for op $\hat{O}$: $\langle\hat{O}\rangle = \frac{\langle\phi|e^{T^\dagger}\hat{O}e^T|\phi\rangle}{\langle\phi|e^{T^\dagger}e^T|\phi\rangle}$
- direct computation requires infinite sum.
- define $\overline{O} = e^{-T}\hat{O}e^T$ and $\langle\tilde{\psi}| = \langle\phi|\frac{e^{T^\dagger}e^T}{\langle\phi|e^{T^\dagger}e^T|\phi\rangle}$
- then $\langle\hat{O}\rangle = \langle\tilde{\psi}|\overline{O}|\phi\rangle$ and
  $\langle\tilde{\psi}'|\phi\rangle = \langle\phi'|\frac{e^{T^\dagger}e^T}{\langle\phi|e^{T^\dagger}e^T|\phi\rangle}|\phi\rangle = \delta(\underline{P}' - \underline{P})$
- $\overline{O}$ computed from Baker–Hausdorff expansion:
  $\overline{O} = \hat{O} + [\hat{O},T] + \frac{1}{2}[[\hat{O},T],T] + \cdots$
- $\langle\tilde{\psi}|$ is a left eigenvector of $\overline{\mathcal{P}^-}$:
  $\langle\tilde{\psi}|\overline{\mathcal{P}^-} = \langle\phi|\frac{e^{T^\dagger}\mathcal{P}^-e^T}{\langle\phi|e^{T^\dagger}e^T|\phi\rangle} = \langle\phi|\overline{\mathcal{P}^-}^\dagger\frac{e^{T^\dagger}e^T}{\langle\phi|e^{T^\dagger}e^T|\phi\rangle} = \frac{M^2+P_\perp^2}{P^+}\langle\tilde{\psi}|$

# LFCC approximation

- the form factor is approximated by the matrix element

$$F_1(q^2) = 8\pi^3 \langle \widetilde{\psi}^{\pm}(\underline{P}+\underline{q}) | \overline{J^+(0)} | \phi^{\pm}(\underline{P}) \rangle,$$

  with $\overline{J^+(0)} = J^+(0) + [J^+(0), T] + \cdots$

- for this model, there are no contributions from fermion-antifermion pairs, so that

$$J^+(0) = 2\sum_s \int \frac{d\underline{p}'}{\sqrt{16\pi^3}} \int \frac{d\underline{p}}{\sqrt{16\pi^3}} b_s^\dagger(\underline{p}') b_s(\underline{p}),$$

- only the first two terms of the Baker–Hausdorff expansion contribute to the matrix element
- the first term contributes $1/8\pi^3$

# The second term

$$[J^+(0),T] \;=\; 2\sum_{ls}\int \frac{d\underline{p}'}{\sqrt{16\pi^3}}\int \frac{d\underline{p}}{\sqrt{16\pi^3}}\int d\underline{q}'[t_{ls}(\underline{q}',\underline{p}) - t_{ls}(\underline{q}',\underline{p}')] \times a_l^\dagger(\underline{q}')b_s^\dagger(\underline{p}')b_s(\underline{p}).$$

$$\langle\widetilde{\psi}^\pm(\underline{P}+\underline{q})|[J^+(0),T]|\phi^\pm(\underline{P})\rangle = \sum_l \frac{(-1)^l}{8\pi^3}\int d\underline{q}'\theta(P^+ + q^+ - q'^+) \times l_{l\pm}^\pm(\underline{q}',\underline{P}+\underline{q})[\theta(P^+ - q'^+)t_{l\pm}(\underline{q}',\underline{P}-\underline{q}') - t_{l\pm}(\underline{q}',\underline{P}+\underline{q}-\underline{q}')],$$

where $l_{l\pm}^\pm$ is the left-hand wave function:

$$\langle\widetilde{\psi}^\sigma(\underline{P})| = \langle\phi^\sigma(\underline{P})| + \sum_{ls}\int d\underline{q}\theta(P^+ - q^+)l_{ls}^{\sigma *}(\underline{q},\underline{P})\langle 0|a_l(\underline{q})b_s(\underline{P}-\underline{q}).$$

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# Evaluation of the form factor

- the left-hand wave function takes the form
$l^\sigma_{ls}(\underline{q},\underline{P}) = \delta_{\sigma s}\frac{-g}{\sqrt{16\pi^3 q^+}}\left(\frac{P^+-q^+}{P^+}\right)^\gamma \frac{q^+/P^+}{\mu_l^2+q_\perp^2}\tilde{l}(q^+/P^+)$
with $\tilde{l}$ the solution of a 1D integral equation
- if $\tilde{l}$ is computed in quadrature, the integrals remaining in $F_1$ can be computed from the same quadrature rule for any chosen value of $q^2$
- if $\tilde{l}$ is instead constructed as an expansion in $g^2$, $F_1$ can also be constructed as an expansion
- in any case, in the limit of $q^2 \to 0$, we have $F_1(0) = 1$, consistent with the unit charge in the current $J^+ = \bar{\psi}\gamma^+\psi$

# Light-front holographic QCD

A factorized wave function in the valence ($q\bar{q}$) sector

$$\psi = e^{iL\varphi} X(x)\phi(\zeta)/\sqrt{2\pi\zeta}$$

subject to an effective potential $\widetilde{U}$ that conserves $L_z$:

$$\left[\frac{\mu_1^2}{x} + \frac{\mu_2^2}{1-x} - \frac{\partial^2}{\partial\zeta^2} - \frac{1-4L^2}{4\zeta^2} + \widetilde{U}\right] X(x)\phi(\zeta) = M^2 X(x)\phi(\zeta).$$

For zero-mass quarks, the longitudinal wave function $X$ decouples, and the transverse wave function satisfies

$$\left[-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U(\zeta)\right]\phi(\zeta) = M^2\phi(\zeta),$$

with $U$ determined by an AdS$_5$ correspondence.

# Softwall model for massless quarks

Yields oscillator potential $U(\zeta) = \kappa^4\zeta^2 + 2\kappa^2(J-1)$.
Spectrum: $M^2 = 4\kappa^2\,(n + (J+L)/2)$
$\rightarrow$ linear Regge trajectory, good fit for light mesons.
Transverse wave functions are 2D oscillator functions.
Longitudinal $X$ constrained by a form-factor duality.

$$F(q^2) = \int \frac{dx\,|X(x)|^2}{x(1-x)} \int 2\pi\zeta d\zeta J_0\left(\zeta q_\perp\sqrt{x/(1-x)}\right)|\phi(\zeta)|^2.$$

To be compared with the form computed in AdS$_5$

$$F(q^2) = \int dx \int 2\pi\zeta d\zeta J_0\left(\zeta q_\perp\sqrt{x/(1-x)}\right)|\phi(\zeta)|^2.$$

Thus $X(x) = \sqrt{x(1-x)}$ when the quarks are massless.

# Massive quarks

*Ansatz* by Brodsky and De Téramond:
replace $k_\perp^2/(x(1-x))$ with $k_\perp^2/x(1-x)+\mu_1^2/x+\mu_2^2/(1-x)$
in transverse harmonic oscillator eigenfunctions,
with $\mu_i$ as current-quark masses

$$X_{\rm BdT}(x)=N_{\rm BdT}\sqrt{x(1-x)}e^{-(\mu_1^2/x+\mu_2^2/(1-x))/2\kappa^2},$$

Instead, use a longitudinal equation with the ‘t Hooft model

$$\left[\frac{m_1^2}{x}+\frac{m_2^2}{1-x}\right]X(x)+\frac{g^2}{\pi}\mathcal{P}\int dy\frac{X(x)-X(y)}{(x-y)^2}-CX(x)=M_\parallel^2X(x),$$

and $m_i$ constituent masses. Then $X(x)$ well approximated by $x^{\beta_1}(1-x)^{\beta_2}$, with $m_i^2\pi/g^2-1+\pi\beta_i\cot\pi\beta_i=0$. Need $\beta_1=\beta_2=1/2$ for zero current masses $\Rightarrow g^2/\pi=m_u^2=m_d^2$.

# Numerical solution

Expand solution as $X(x)=\sum_n c_n f_n(x)$ with respect to basis functions

$$f_n(x)=N_n x^{\beta_1}(1-x)^{\beta_2}P_n^{(2\beta_2,2\beta_1)}(2x-1);$$

the $n=0$ term represents 90% or more of the probability.
Matrix representation of longitudinal equation, for $M_{||}=0$,

$$\left(\frac{m_1^2}{m_u^2}A_1+\frac{m_2^2}{m_u^2}A_2+B\right)\vec{c}=\xi\vec{c}, \quad \text{with } \xi\equiv C/m_u^2 \text{ and}$$

$$\begin{aligned}(A_1)_{nm} &= \int_0^1 \frac{dx}{x} f_n(x)f_m(x), \quad (A_2)_{nm}=\int_0^1 \frac{dx}{1-x} f_n(x)f_m(x), \\ B_{nm} &= \frac{1}{2}\int_0^1 dx \int_0^1 dy \frac{f_n(x)-f_n(y)}{x-y}\frac{f_m(x)-f_m(y)}{x-y}.\end{aligned}$$

# Parameters and decay constants

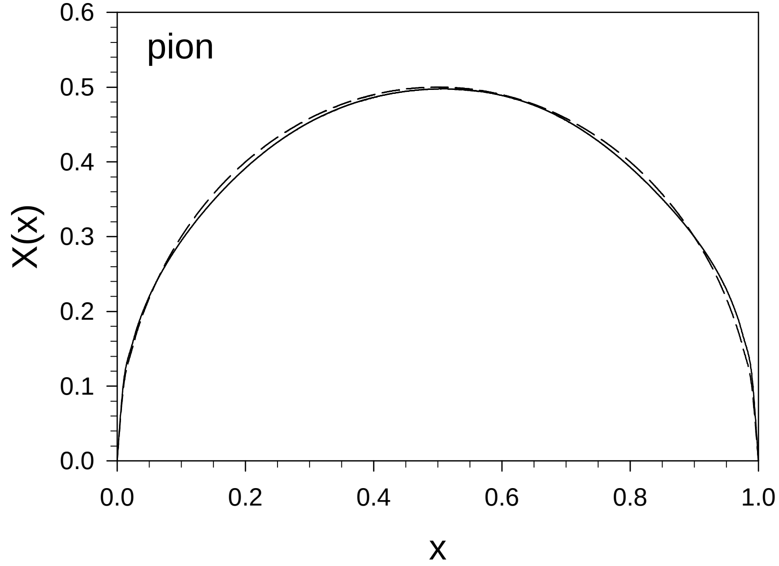
| | model | | *ansatz* | | | | decay constant | | |
|---|---|---|---|---|---|---|---|---|---|
| meson | $m_1$ | $m_2$ | $\mu_1$ | $\mu_2$ | $P_{q\bar{q}}$ | $\kappa$ | model | *ansatz* | exper. |
| pion | 330 | 330 | 4 | 4 | 0.204 | 951 | 131 | 132 | 130 |
| kaon | 330 | 500 | 4 | 101 | 1 | 524 | 160 | 162 | 156 |
| J/$\Psi$ | 1500 | 1500 | 1270 | 1270 | 1 | 894 | 267 | 238 | 278 |

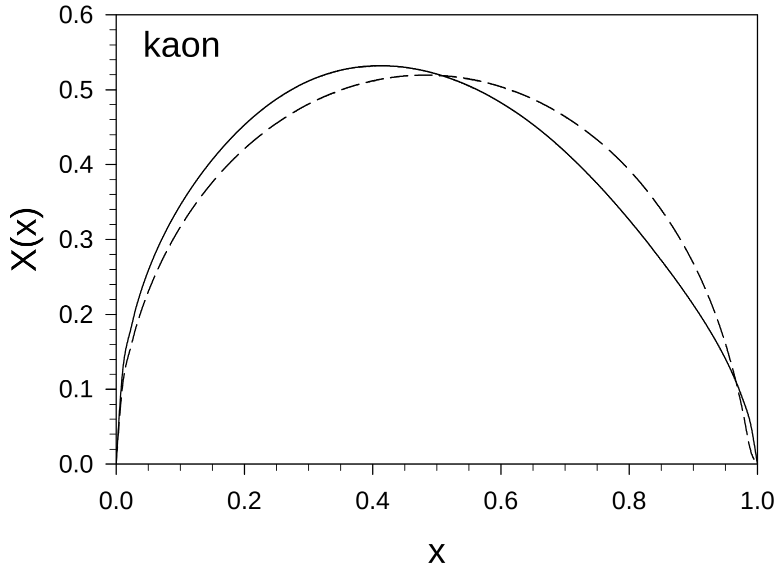
All dimensionful parameters are in units of MeV. Parameter and experimental values are from Vega *et al*., PRD **80**, 055014 (2009) and the Particle Data Group.

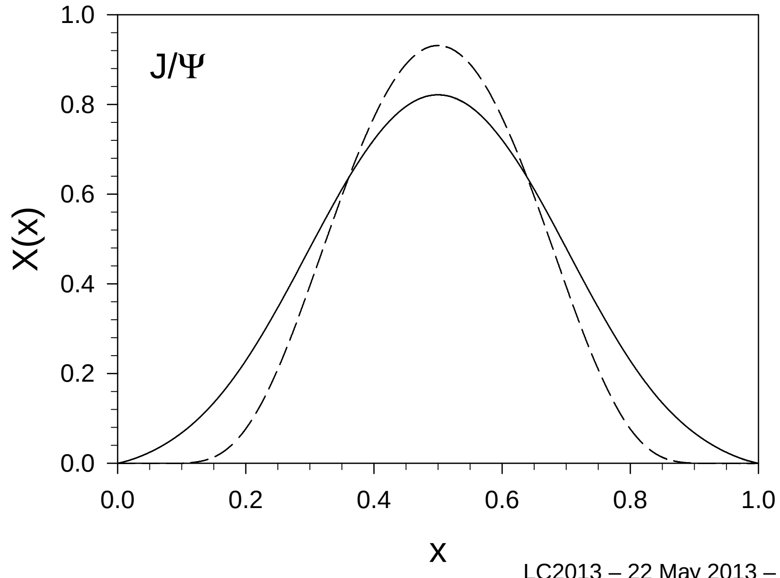
The decay constant is given by

$$f_M = 2\sqrt{6}\int_0^1 dx \int_0^\infty \frac{dk_\perp^2}{16\pi^2}\psi(x,k_\perp).$$

# Longitudinal wave functions

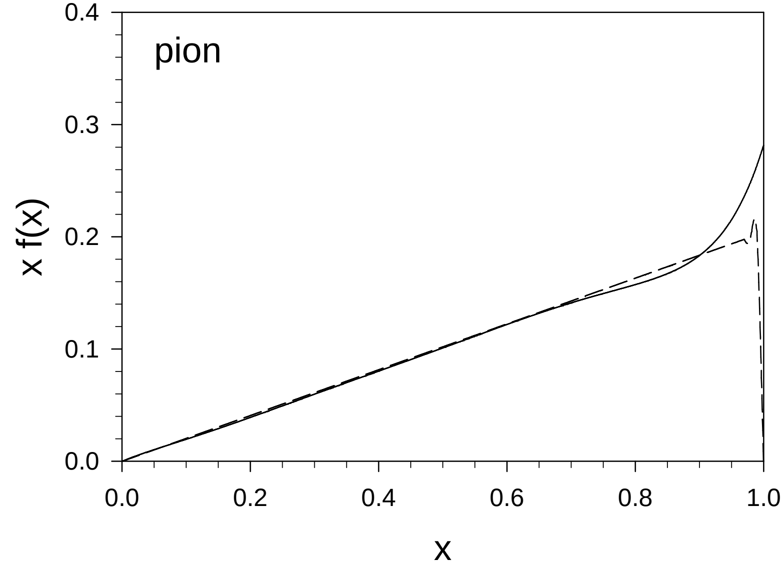


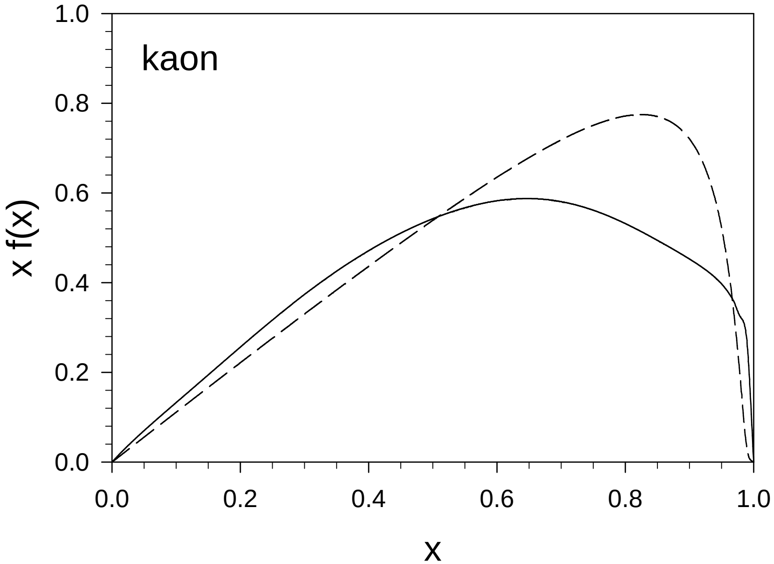



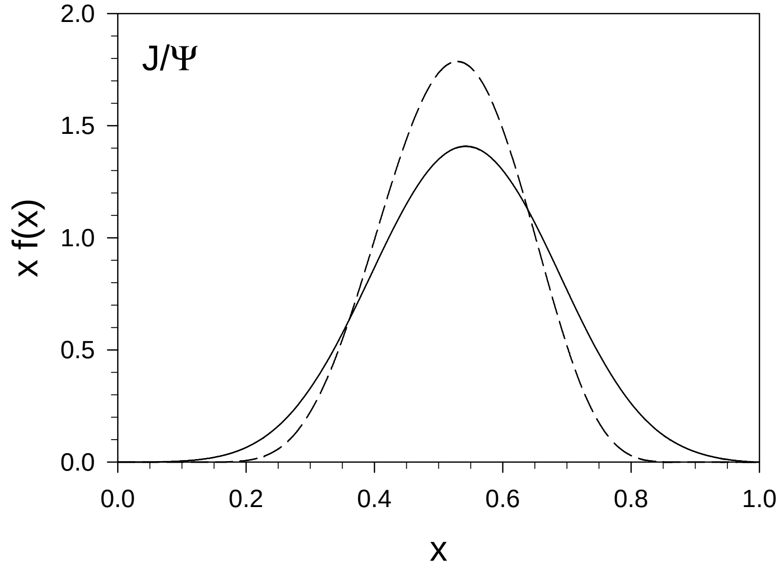



# Parton distributions

$$f(x) = P_{q\bar{q}} \frac{X^2(x)}{x(1-x)}$$



# Summary

- LFCC method
  - divides hadronic eigenproblem into
    - eigenproblem in valence sector.
    - auxiliary equations to define.
- advantages of LFCC
  - no Fock-space truncation.
  - no sector dependence or spectator dependence.
  - systematically improvable.
- light-front holography
  - provides model for valence sector.
  - can augment for longitudinal wave functions, consistent with Brodsky-de Téramond *ansatz*.
  - leads to natural choice for basis functions, applicable beyond holographic approximation.