Evolution of the light-like Wilson polygons

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Fundamental .be Interactions

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LIGHTCONE 2013$^+$
Venturing off the lightcone - local vs. global features

20-24 May 2013, Skiathos, Greece

Ancient Greece

Aristotle: three divisions of human intellectual activity

- **physics**: studies the causes of change of material things
- **mathematics**: addresses abstract quantity
- **metaphysics**: concerned with being as such

Ergo: physics is the learning of **evolution**

Many Faces of QCD Evolution

Renaissance: **3D structure of the nucleon**

Many Faces of QCD Evolution

Renaissance: **3D structure of the nucleon**

- Experiment
- Phenomenology
- Theory

Many Faces of QCD Evolution

Renaissance: **3D structure of the nucleon**

- Experiment:
  - **SIDIS process** $lH^{\uparrow} \to l' h X$**:** HERMES, COMPASS, JLab 12 GeV
  - **DY process** $H_1^{(\uparrow)} H_2^{\uparrow} \to l^+ l^- X$ **:** COMPASS, PAX, GSI, RHIC
  - **Hadron collisions** $H_1^{(\uparrow)} H_2^{\uparrow} \to l^+ l^- X$ **:** RHIC
  - $e^+ e^- \to h_1 h_2 X$**:** BELLE, BaBar
  - future **EIC**
- Phenomenology: see, e.g., talks by **Efremov, Pasquini**
- Theory: see, e.g., talks by **Mulders, Vladimirov**

Many Faces of QCD Evolution

Renaissance: **3D structure of the nucleon**

- Experiment
- Phenomenology
- Theory
- Mathematics

Many Faces of QCD Evolution

- Experiment
- Phenomenology
- Theory
- Mathematics: **Loop Space** analysis
  - Chen iterated line integrals (via the holonomy of a connection in a principal fiber bundle)
  - Group of loops and generalised group of loops
  - Loop calculus: endpoint and area derivatives; variational derivative
  - Connection with knot theory *etc.*

@ [Chen (1968, 1971, 1973); Gambini, Trias (1980, 1981, 1986); Tavares (1993)]

Gauge-Invariant Hadronic Correlators on the Light-Cone

$$\Phi(k^+) = \text{F.T.} \ \langle h| \ \bar{\Psi}(z^-) \ \mathcal{W}_{n^-}[z^-, 0^-] \ \Psi(0^-) \ |h\rangle$$

Gauge invariance is guaranteed by the **light-like Wilson line**

$$\mathcal{W}\,[y, x]_w^\Gamma = \mathcal{P} \exp \left[-ig \int_\Gamma d\tau \, w^\mu \mathcal{A}_\mu^a(w\tau) \right]$$

Saving gauge invariance, we get **path-dependence**: very important!

"Animal Farm" rule for the field correlators: [almost] all correlators are singular, but those on the light-cone are [expected to be] more singular than others. The same is true for the Wilson lines/loops.

3D Correlators

Generic 3D correlator with the light-like and transverse gauge links

@ [Belitsky, Ji, Yuan (2003); Boer, Mulders, Pijlman (2003)]

$$\Phi(k^+, k_\perp; \mathrm{scales}) \sim$$
$$\mathrm{F.T.}\ \langle h|\bar{\psi}(z)\ \mathcal{W}_{n\cup l_\perp}[z^-, z_\perp; 0^-, 0_\perp]\psi(0)|h\rangle$$

Tree-level:

$$\Phi^{(0)}(k^+, k_\perp) = \delta(k^+ - p^+)\delta^{(2)}(k_\perp)$$

$$\int d^2k_\perp \Phi(k^+, k_\perp) = \Phi(k^+) = \mathrm{collinear\ limit}$$

$$\Phi(k^+, \mu) = \mathrm{F.T.}\ \langle h|\bar{\psi}(z)\ \mathcal{W}_n[z^-, 0^-]\psi(0)|h\rangle$$

One-loop corrections: $\to$ emergent (light-cone/rapidity) singularities!

Classification of Singularities

- Ultraviolet poles $\sim \frac{1}{\varepsilon}$
- Overlapping divergences: contain the UV and rapidity poles simultaneously $\sim \frac{1}{\varepsilon} \ \ln\theta$
- Pure rapidity divergences: $\sim \ln^{1,2}\theta$:
- Specific self-energy divergences: stem from the gauge links, treated by modifications of the soft factors

@ [ICh, Stefanis (2008, 2009, 2010); Collins (2003, 2008, 2011, 2012 etc.); Chiu, Jain, Neill, Rothstein (2011, 2012); Avsar (2012) Idilbi, Scimemi (2011, 2012)]

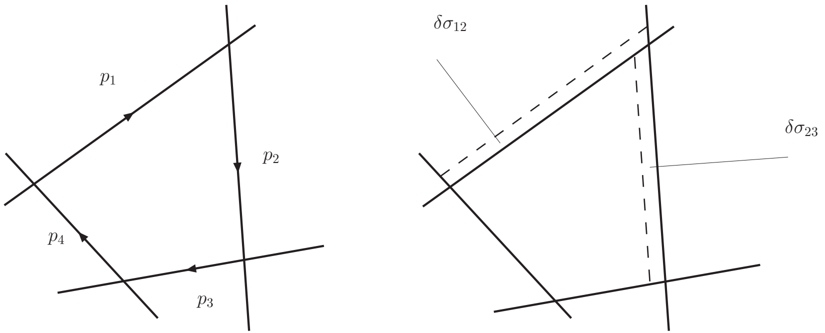
- Penetration of the extra singularities in the anomalous dimensions of the TMDs:

@ [ICh, Stefanis (2008, 2009, 2010)]

- Collinear case: cancellation in the interplay of the virtual and real gluon contributions:

@ [Furmanski, Curci, Petronzio (1980); Fleming, Zhang (2012)]

Singularities of Light-Like Cusped Wilson Loops
Generic light-like quadrilateral contour



@ [Alday, Maldacena (2007); Makeenko (2003); Korchemsky, Drummond, Sokatchev (2008); Alday et al. (2011); Beisert et al. (2012); Belitsky (2012)]

Motivation: **duality** between 4-gluon planar scattering amplitude in $\mathcal{N} = 4$ SYM and the Wilson loop made up from four light-like segments:

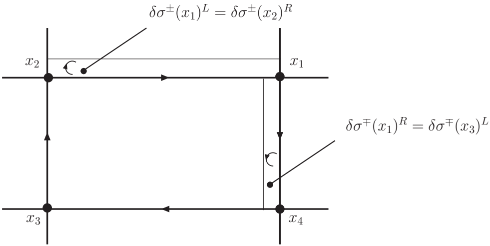
$$x_i - x_{i+1} = p_i$$

are equal to the external momenta of this 4-gluon amplitude. The IR evolution of the former is dual to the UV evolution of the latter: governed by the cusp anomalous dimension.

@ [Korchemsky, Radyushkin (1987)]

Singularities of Light-like Cusped Wilson Loops
Planar rectangular contour

@ [Korchemskaya, Korchemsky (1992); Bassetto, Korchemskaya, Korchemsky, Nardelli (1993)]

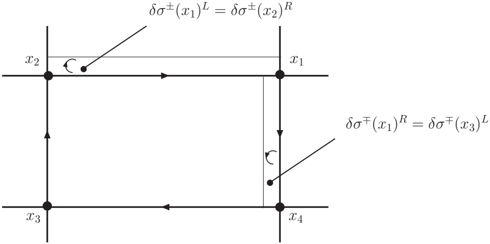


$$\sigma \equiv 2N^+N^- \ , \ \lim_{N_c \to \infty} \mathcal{W}[\Gamma] =$$

$$1 + \frac{\alpha_s N_c}{2\pi} \left\{ - \frac{1}{\epsilon^2} \left(\left[-\sigma\mu^2 + i0 \right]^{\epsilon} + \left[\sigma\mu^2 + i0 \right]^{\epsilon} \right) + \text{finite} \right\} + O(\alpha_s N_c)$$

Singularities of Light-like Cusped Wilson Loops

Planar rectangular contour



– $\mathcal{W}[\Gamma]$ is not renormalizable due to light-cone extra divergences
– Aristotle: physics concerns with evolution, not the Wilson loops as such
– area logarithmic derivative decreases the power of divergency

$$\frac{d \ln \mathcal{W}[\Gamma]}{d \ln \sigma} = -\frac{\alpha_s N_c}{2\pi} \, \frac{1}{\epsilon} \left([\sigma\mu^2 + i0]^\epsilon - [-\sigma\mu^2 + i0]^\epsilon \right)$$

Anomalous dimension results from (large-N_c):

$$\mu \frac{d}{d\mu} \frac{d\ \mathcal{W}[\Gamma]}{d \ln \sigma} \sim -4\ \Gamma_{\rm cusp} \cdot \mathcal{W}[\Gamma]\ ,\ \Gamma_{\rm cusp} = \frac{\alpha_s N_c}{2\pi}$$

We get finite result by means of the area derivative: **dynamical properties** of the light-like Wilson loop are encoded in the cusp anomalous dimension

@ [Korchemsky, Radyushkin (1987)]

Local quantity: behavior in vicinity of an obstruction. Path-dependence shows up in finite terms. We related the **geometry** of the loop space (area differentials) and the **dynamics** of the fundamental d.o.f., that is the light-like Wilson loops.

@ [Polyakov (1979); Makeenko, Migdal (1979, 1981); Kazakov, Kostov (1980); Brandt et al. (1981, 1982); Stefanis et al. (1989, 2003)]

Wilson loops as the (fundamental) gauge-invariant degrees of freedom:

$$\mathcal{W}_n[\Gamma_1, ...\Gamma_n] = \langle 0|\mathcal{T}\frac{1}{N_c}\Phi(\Gamma_1)\cdots\frac{1}{N_c}\Phi(\Gamma_n)|0\rangle$$

$$\Phi(\Gamma_i) = \mathcal{P}\ \exp\left[ig\int_{\Gamma_i} dz^\mu \mathcal{A}_\mu(z)\right]$$

The Wilson functionals obey the Makeenko-Migdal loop equations:

$$\partial^\nu\ \frac{\delta}{\delta\sigma_{\mu\nu}(x)}\ \mathcal{W}_1[\Gamma] = N_c g^2\ \oint_\Gamma\ dz^\mu\ \delta^{(4)}(x-z)\mathcal{W}_2[\Gamma_{xz}\Gamma_{zx}]$$

$+$ Mandelstam contraints

$$\sum a_i \mathcal{W}_{n_i}[C_1^i ... C_{n_i}^i] = 0$$

The equation is exact and non-perturbative, but not closed and difficult to solve in general.

Loop Space
Makeenko-Migdal approach

Area derivative:

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\Phi(\Gamma) = \lim_{|\delta\sigma_{\mu\nu}(x)|\to 0} \frac{\Phi(\Gamma\delta\Gamma_x) - \Phi(\Gamma)}{|\delta\sigma_{\mu\nu}(x)|}$$

Path derivative:

$$\partial_\mu \Phi(\Gamma) = \lim_{|\delta x_\mu|\to 0} \frac{\Phi(\delta x_\mu^{-1}\Gamma\delta x_\mu) - \Phi(\Gamma)}{|\delta x_\mu|}$$

Mandelstam formula:

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\mathrm{Tr}\ \Phi(\Gamma) = ig\mathrm{Tr}\ [F_{\mu\nu}\Phi(\Gamma)]$$

However:

- No information about cusps and other obstructions
- Wilson loops are functionals defined on the paths. But infinitesimal variation of a path doesn't necessarily yield infinitesimal variation of a functional

  @ [ICh, Mertens (2013) to appear]
- Variational analysis in the loop space is by no means straightforward

Loop Space
Makeenko-Migdal approach

$$\mathcal{W}[\Gamma] = \mathcal{W}^{(0)} + \mathcal{W}^{(1)} = 1 - \frac{g^2 C_F}{2} \oint_\Gamma \oint_\Gamma dz_\mu dz'_\nu \ D^{\mu\nu}(z - z') + O(g^4)$$

$$D^{\mu\nu}(z - z') = -g^{\mu\nu} \ \Delta(z - z')$$

$$\Delta(z - z') = \frac{\Gamma(1-\epsilon)}{4\pi^2} \frac{(\pi\mu^2)^\epsilon}{[-(z-z')^2 + i0]^{1-\epsilon}}$$

$$\frac{\delta \ \mathcal{W}[\Gamma]}{\delta\sigma_{\mu\nu}} == \frac{g^2 C_F}{2} \frac{\delta}{\delta\sigma_{\mu\nu}} \oint_\Gamma \oint_\Gamma dz_\lambda dz'^\lambda \ \Delta(z - z') + O(g^4)$$

Use the Stokes theorem

$$\oint_\Gamma dz_\lambda\ \mathcal{O}^\lambda = \frac{1}{2}\int_\Sigma\ d\sigma_{\lambda\rho}(\partial^\lambda\mathcal{O}^\rho - \partial^\rho\mathcal{O}^\lambda)\ ,\ \mathcal{O}^\lambda = \oint_\Gamma dz^\lambda\ \Delta(z)$$

$$\partial_\mu \frac{\delta\ \mathcal{W}[\Gamma_{\text{smooth}}]}{\delta\sigma_{\mu\nu}(x)} = \frac{g^2 N_c}{2}\oint_{\Gamma_{\text{smooth}}} dy^\nu\ \delta^{(\omega)}(x-y) + O(g^4)$$

Example: 2D QCD

$$\mathcal{W}[\Gamma_{\text{smooth}}]^{\text{2D}} = \exp\left[-\frac{g^2 N_c}{2}\Sigma\right]\ ,\ \Sigma = \text{ area inside } \Gamma_{\text{smooth}}$$

Problems:

- Most interesting loops are divergent and have obstructions: we are particularly interested in **cusped** loops. In that case, renormalized version of the MM equation is not available
- The area functional derivative is not well-defined operation for arbitrary contour. In particular, the area differentiation for cusped loops is not (at least) straightforward
- Problems with continuous deformation of the loops in the Minkowski space: consistent definition of the derivatives obscure
- Connection of the loop functionals to observables
- Solution of the MM equations in the four-dimensional Minkowskian space-time is not known

Simplifications:

- Large-N_c limit: factorization property
  $W_2(C_1, C_2) \approx W_1(C_1) \cdot W_2(C_2)$
- Null-plane light-cone rectangular contours are effectively two-dimensional
- Light-like polygons with conserved angles: no angle-dependent contributions which may break MM-equation
- Area differentiation: the power of divergency decreases

Therefore, the MM approach relates cusp dynamics, renormalization properties and geometry of the loop space. The problem now is how to extract reliable information.

Loop Space
Schwinger approach
@ [Schwinger (1951)]

Fundamental quantum dynamical principle

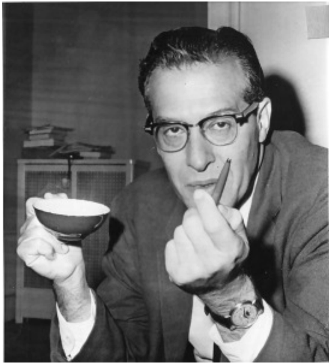
$$\delta\langle\ '\ |\ ''\ \rangle = \frac{i}{\hbar}\langle\ '\ |\delta S|\ ''\ \rangle$$

Application to the Wilson functionals $\Phi(\Gamma)$ + Mandelstam formula + Stokes theorem yields MM Eq.

$$\langle 0|\nabla_\mu F^{\mu\nu}\ \mathrm{Tr}\Phi(\Gamma)|0\rangle = i\hbar\langle 0|\frac{\delta}{\delta A_\nu}\ \mathrm{Tr}\Phi(\Gamma)|0\rangle$$

$$\partial^\nu\ \frac{\delta}{\delta\sigma_{\mu\nu}(x)}\ \mathcal{W}_1(\Gamma) = N_c g^2\ \oint_\Gamma\ dz^\mu\ \delta^{(4)}(x-z)\mathcal{W}_2(\Gamma_{xz}\Gamma_{zx})$$

Loop Space
Schwinger warning



Did you take care of singularities?

Loop Space

Shape variations without Stokes theorem

$$\mathcal{W}^{(1)}[\Gamma_\Box] = \frac{g^2 C_F}{2} \frac{\Gamma(1-\epsilon)(\pi\mu^2)^\epsilon}{4\pi^2} \cdot$$

$$\sum_{i,j} (v_j^\lambda v_j^\lambda) \cdot \int_0^1 \int_0^1 \frac{d\tau d\tau'}{[-(x_i - x_j - \tau_i v_i + \tau_j v_j)^2 + i0]^{1-\epsilon}}$$

$$2(v_1 v_2) = 2N^+ N^- \ , \ \frac{\delta}{\delta \ln \sigma} \equiv \sigma_{+-} \frac{\delta}{\delta \sigma_{+-}} + \sigma_{-+} \frac{\delta}{\delta \sigma_{-+}}$$

$$\frac{\delta \ \mathcal{W}[\Gamma_\Box]}{\delta \sigma_{\mu\nu}} = - \frac{\alpha_s N_c}{2\pi} \Gamma(1-\epsilon)(\pi\mu^2)^\epsilon \frac{\delta}{\delta \sigma_{\mu\nu}} (-2N^+ N^-)^\epsilon \ \frac{1}{2} \int_0^1 \int_0^1 \frac{d\tau d\tau'}{[(1-\tau)\tau']^{1-\epsilon}}$$

$$\mu \frac{d}{d\mu} \left[\frac{\delta}{\delta \ln \sigma} \ \ln \ \mathcal{W}[\Gamma_\Box] \right] = - \sum \Gamma_{\rm cusp}$$

Loop Space

Schwinger approach for light-like planar contours: Effective quantum action?

Return to the definition of the area derivative and consider special area differentials (do not make use of the Stokes theorem or Mandelstam formula)

Logarithmic derivative in the renormalization scale allows one to obtain the finite result

$$\mu \frac{d}{d\mu} \left[\sigma_{\mu\nu} \frac{\delta}{\delta \sigma_{\mu\nu}} \mathcal{W}[\Gamma] \right] = - \sum \Gamma_{\text{cusp}} \cdot \mathcal{W}[\Gamma]$$

– works for the rectangular and Π-shape light-like planar contours; 3D correlators "on the light-cone"?

– complete evolution of the 3D functions + further development...

– effective quantum action $S_{\text{eft}} = S_{\text{eff}}[\Gamma_{\text{cusp}}]$ takes into account **geometric properties of the paths** in the loop space: in progress

Outlook I:

- Makeenko-Migdal approach provides a full and consistent description of the geometrical properties of the loop space. Fundamental degrees of freedom are closed Wilson loops and the MM Eqs. resemble the Schwinger-Dyson Eqs. in the loop space. In general, the system of the MM Eqs. is not closed and cannot be straightforwardly applied to calculate any useful quantity.

- However, in the large-N_c limit, in the null-plane $z_\perp = 0$, for the rectangular planar light-like Wilson loops, the area functional derivative is reduced to the normal derivative for the dimensionally regularized (not renormalized!) loops and the MM Eqs. appear to be equivalent to the energy/rapidity evolution equations.

Outlook II:

- Geometrical properties of the Wilson loop space provide a hint for understanding singularities and evolution of gauge invariant quantum field correlators with light-like and off-light-cone Wilson lines and loops with cusps and self-intersectons (collinear PDFs, TMDs, high-energy amplitudes, heavy quarks, etc.)
- To relate geometrical properties of the loop space and the dynamics encoded in cusps is a challenge. The cusps are introduced by externally-driven obstructions of (initially) smooth Wilson loops.
- Conjecture: since the quantum dynamical Schwinger approach is universal, it can be applied to construction of the energy/rapidity evolution equations in many interesting situations. Specific properties of the Wilson loops are determined by the contours with cusps (and/or self-intersections).

Technical details:

1. I.O. Cherednikov,
"On singularities of the TMDs, their origin and treatment"
Nuovo Cim. C036 (2013) *[in print]*

2. I.O. Cherednikov, T. Mertens, F.F. Van der Veken,
"Cusped light-like Wilson loops in gauge theories"
Phys. Part. Nucl. 44 (2013) 250; `arXiv:1210.1767 [hep-ph]`

3. F.F. Van der Veken, I.O. Cherednikov, T. Mertens,
"Evolution and dynamics of cusped light-like Wilson loops in loop space"
AIP Conf. Proc. 1523 (2013) 272; `arXiv:1212.4345 [hep-th]`

4. I.O. Cherednikov, T. Mertens, F.F. Van der Veken,
"Loop space and evolution of the light-like Wilson polygons"
Int. J. Mod. Phys. Conf. Ser. 20 (2012) 109; `arXiv:1208.5410 [hep-th]`

5. I.O. Cherednikov, T. Mertens, F.F. Van der Veken,
"Evolution of cusped light-like Wilson loops and geometry of the loop space"
Phys. Rev. D86 (2012) 085035; `arXiv:1208.1631 [hep-th]`

6. I.O. Cherednikov,
"Layout of Wilson lines and light-cone peculiarities of transverse-momentum dependent PDFs"
PoS (**QNP**2012) 061; `arXiv:1206.4212 [hep-ph]`