Scalar theories and symmetry breaking in the light-front coupled-cluster method

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Objectives

- include zero modes in the light-front coupled-cluster method, to facilitate analysis of theories with symmetry breaking.
- test with applications to ϕ^3 and ϕ^4 theories in two dimensions.
- compare with variational coherent-state analyses.

LFCC method: Phys Lett B **711**, 417 (2012)

To solve $\mathcal{P}^-|\psi\rangle = \frac{M^2+P_\perp^2}{P^+}|\psi\rangle$ without truncation,
build eigenstate as $|\psi\rangle = \sqrt{Z}e^T|\phi\rangle$ from valence state $|\phi\rangle$
and operator T that increases particle number:

$$e^{-T}\mathcal{P}^-e^T|\phi\rangle = e^{-T}\frac{M^2+P_\perp^2}{P^+}e^T|\phi\rangle,$$

New effective Hamiltonian $\overline{\mathcal{P}^-} = e^{-T}\mathcal{P}^-e^T$,
using a Baker–Hausdorff expansion

$$\overline{\mathcal{P}^-} = \mathcal{P}^- + [\mathcal{P}^-, T] + \frac{1}{2}[[\mathcal{P}^-, T], T] + \dots$$

Eigenvalue problem becomes $\overline{\mathcal{P}^-}|\phi\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi\rangle$
Project it onto the valence and orthogonal sectors

$$P_v\overline{\mathcal{P}^-}|\phi\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi\rangle, \qquad (1-P_v)\overline{\mathcal{P}^-}|\phi\rangle = 0.$$

No spectator dependence and no uncanceled divergences!

ϕ^3 theory: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{3!}\phi^3$

The mode expansion for the field at zero light-front time is

$$\phi = \int \frac{dp^+}{\sqrt{4\pi p^+}} \left\{ a(p^+) e^{-ip^+x^-/2} + a^\dagger(p^+) e^{ip^+x^-/2} \right\},$$

with the modes quantized such that

$$[a(p^+), a^\dagger(p'^+)] = \delta(p^+ - p'^+).$$

The normal-ordered light-front Hamiltonian $\mathcal{P}^- = \mathcal{P}^-_{\rm free} + \mathcal{P}^-_{\rm int}$ is

$$\begin{aligned} \mathcal{P}^-_{\rm free} &= \int dp^+ \frac{\mu^2}{p^+} a^\dagger(p^+) a(p^+) \\ &+ \frac{\mu^2}{2} \int \frac{dp_1^+ dp_2^+}{\sqrt{p_1^+ p_2^+}} \delta(p_1^+ + p_2^+) \left[a^\dagger(p_1^+) a^\dagger(p_2^+) + a(p_1^+) a(p_2^+) \right], \end{aligned}$$

with zero-mode terms included here and in $\mathcal{P}^-_{\rm int}$.

$\phi^3\ \mathcal{P}^-_{\text{int}}$

The interaction term is

$$\mathcal{P}^-_{\text{int}} = \frac{\lambda}{2}\int \frac{dp^+ dp'^+}{\sqrt{4\pi p^+ p'^+ (p^+ - p'^+)}} \Big[a^\dagger(p^+) a(p'^+) a(p^+ - p'^+)$$

$$+ a^\dagger(p'^+) a^\dagger(p^+ - p'^+) a(p^+)\Big]$$

$$+ \frac{\lambda}{6}\int \frac{dp_1^+ dp_2^+ dp_3^+}{\sqrt{4\pi p_1^+ p_2^+ p_3^+}} \delta(p_1^+ + p_2^+ + p_3^+)$$

$$\times \Big[a^\dagger(p_1^+) a^\dagger(p_2^+) a^\dagger(p_3^+) + a(p_1^+) a(p_2^+) a(p_3^+)\Big]$$

T operator

Simplest approximation for T is a single zero-mode creation

$$T = \int_0^\infty dp^+ \sqrt{4\pi p^+} g(p^+) a^\dagger(p^+),$$

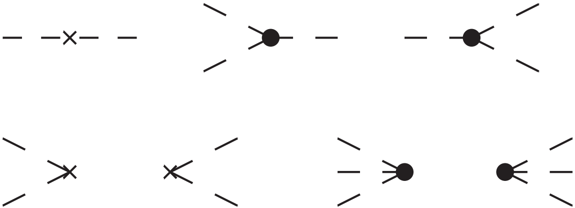
with $g(p^+)$ having support only at $p^+ = 0$ in an appropriate limit. This limit, taken at the end of the calculation, restores momentum conservation. The valence state is the bare vacuum. The projection $1 - P_v$ is truncated to include only states with one zero mode. The corresponding transformation of the field is

$$e^{-T}\phi e^T = \phi + [\phi, T] = \phi + \int dp^+ g(p^+) e^{-ip^+x^-/2},$$

which provides for a constant shift in the limit that $g(p^+) \propto \delta(p^+)$.

Graphical $\mathcal{P}^-$ and T

$\mathcal{P}^-$



T

Lines on the right $\rightarrow$ annihilation ops, on the left $\rightarrow$ creation ops,
the cross $\rightarrow$ kinetic energy contribution,
the dot $\rightarrow$ an interaction.

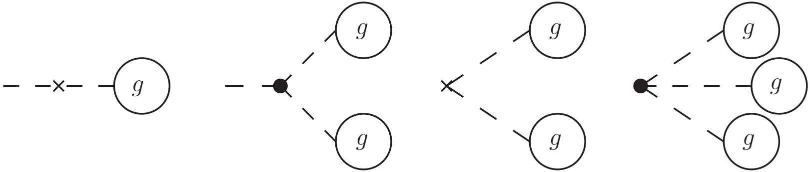
Effective Hamiltonian

Compute effective Hamiltonian $\overline{\mathcal{P}^-}$ from the Baker–Hausdorff expansion:

$$
\begin{aligned}
\overline{\mathcal{P}^-} &= \sqrt{4\pi}\mu^2 \int dp^+ \frac{g(p^+)}{\sqrt{p^+}} a^\dagger(p^+) \\
&+ \frac{1}{2!} 4\pi\mu^2 \int dp_1^+ dp_2^+ \delta(p_1^+ + p_2^+) g(p_1^+) g(p_2^+) \\
&+ \frac{1}{2!}\sqrt{4\pi}\lambda \int \frac{dp^+ dp'^+}{\sqrt{p^+}} g(p'^+) g(p^+ - p'^+) a^\dagger(p^+) \\
&+ \frac{1}{3!} 4\pi\lambda \int dp_1^+ dp_2^+ dp_3^+ \delta(p_1^+ + p_2^+ + p_3^+) g(p_1^+) g(p_2^+) g(p_3^+),
\end{aligned}
$$

keeping only terms that do not annihilate the vacuum and create at most one zero mode!

Graphical effective Hamiltonian



Vacuum state

The eigenvalue problem in the valence (vacuum $|0\rangle$) sector, $P_v\overline{\mathcal{P}^-}|0\rangle = P^-|0\rangle$, is

$$\left[\frac{1}{2}4\pi\mu^2\int dp_1^+dp_2^+\delta(p_1^+ + p_2^+)g(p_1^+)g(p_2^+)\right.$$

$$\left.+\frac{1}{6}4\pi\lambda\int dp_1^+dp_2^+dp_3^+\delta(p_1^+ + p_2^+ + p_3^+)g(p_1^+)g(p_2^+)g(p_3^+)\right]|0\rangle = P^-|0\rangle$$

For a function $g(p^+) = \alpha\delta(p^+)$, the eigenvalue P^- is

$$P^- = \frac{1}{2}\mu^2\alpha^2 4\pi\delta(0) + \frac{1}{6}\lambda\alpha^3 4\pi\delta(0),$$

proportional to the volume

$$\int dx^- = \lim_{p^+\to 0}\int dx^- e^{ip^+x^-/2} = 4\pi\delta(0).$$

Write $P^- = \mathcal{E}^-\int dx^-$ in terms of an energy density $\mathcal{E}^- = \frac{1}{2}\mu^2\alpha^2 + \frac{1}{6}\lambda\alpha^3$. The spectrum is unbounded from below as α goes to negative infinity.

Auxiliary equation

The function g is determined by the auxiliary equation $(1-P_v)\overline{\mathcal{P}^-}|\phi(\underline{P})\rangle = 0$, truncated to only one zero mode,

$$\sqrt{4\pi}\mu^2 \frac{g(p^+)}{\sqrt{p^+}} + \frac{1}{2}\sqrt{4\pi}\lambda \int_0^{p^+} \frac{dp'^+}{\sqrt{p^+}} g(p'^+) g(p^+ - p'^+) = 0$$

Multiply by $\sqrt{p^+}$ and take Laplace transform $G(s) \equiv \int_0^\infty e^{-sp^+} g(p^+) dp^+$ to obtain

$$\mu^2 G(s) + \frac{1}{2}\lambda G(s)^2 = 0,$$

The possible solutions are $G(s)=0$ and $-2\mu^2/\lambda$. Because the inverse transform of a constant is a delta function, we obtain the expected $g(p^+) = \alpha\delta(p^+)$ with $\alpha = 0$ and $\alpha = -2\mu^2/\lambda$. These are the local extrema of $\mathcal{E}^-$; the auxiliary equation does miss the global extrema at $\pm\infty$.

Coherent-state analysis

With the T operator truncated to one zero mode,
$|\alpha\rangle \equiv \sqrt{Z_\alpha} e^T |0\rangle$ is a coherent state.
Can then minimize the vacuum energy density $\langle\alpha|\mathcal{H}|\alpha\rangle$ with respect to α, given the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{3!}\phi^3.$$

The T operator can be written

$$T = \alpha \int_0^\infty dp^+ \sqrt{4\pi p^+} \Delta(p^+) a^\dagger(p^+)$$

where $g(p^+) = \alpha\Delta(p^+)$ and, when a specific form is needed $\Delta(p^+) = \frac{1}{\epsilon} e^{-p^+/\epsilon}$ defined so that $\lim_{\epsilon\to 0} \Delta(p^+) = \delta(p^+)$ for integrals from 0 to ∞.

Vacuum energy density

From the commutators

$$\begin{aligned} [T^\dagger, T] &= 4\pi\alpha^2 \int dp^+ p^+ \Delta^2(p^+) \to \pi\alpha^2, \\ [\phi, T] &= \alpha \int dp^+ \Delta(p^+) e^{-ip^+x^-/2} \to \alpha. \\ [\phi, T^\dagger] &= \alpha \int dp^+ \Delta(p^+) e^{+ip^+x^-/2} \to \alpha, \end{aligned}$$

we have, for real α, the normalization $\sqrt{Z_\alpha} = e^{-\pi\alpha^2/2}$, as well as $\phi|\alpha\rangle = \alpha|\alpha\rangle$, $\langle\alpha|\phi = \langle\alpha|\alpha$, and

$$\langle:\mathcal{H}:\rangle = \frac{1}{2}\mu^2\alpha^2 + \frac{1}{6}\lambda\alpha^3 = \mathcal{E}^-.$$

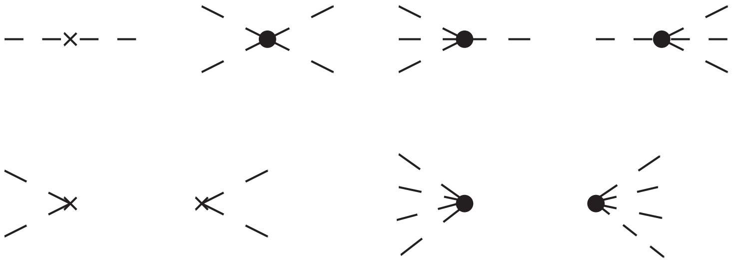
The local extrema are at $\alpha = 0$ and $\alpha = -2\mu^2/\lambda$, and the global extrema at $\pm\infty$, as in the LFCC analysis. The vacuum expectation value for the field is just $\langle\alpha|\phi|\alpha\rangle = \alpha$.

ϕ^4 theory: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4$

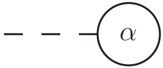
$$
\begin{aligned}
\mathcal{P}^-_{\rm int} \; = \; & \frac{\lambda}{6}\int \frac{dp_1^+ dp_2^+ dp_3^+}{4\pi\sqrt{p_1^+ p_2^+ p_3^+ (p_1^+ + p_2^+ + p_3^+)}} \\
& \qquad \times \Big[a^\dagger(p_1^+ + p_2^+ + p_3^+) a(p_1^+) a(p_2^+) a(p_3^+) \\
& \qquad\qquad + a^\dagger(p_1^+) a^\dagger(p_2^+) a^\dagger(p_3^+) a(p_1^+ + p_2^+ + p_3^+)\Big] \\
& + \frac{\lambda}{4}\int \frac{dp_1^+ dp_2^+}{4\pi\sqrt{p_1^+ p_2^+}} \int \frac{dp_1'^+ dp_2'^+}{\sqrt{p_1'^+ p_2'^+}} \delta(p_1^+ + p_2^+ - p_1'^+ - p_2'^+) \\
& \qquad \times a^\dagger(p_1^+) a^\dagger(p_2^+) a(p_1'^+) a(p_2'^+) \\
& + \frac{\lambda}{24}\int \frac{dp_1^+ dp_2^+ dp_3^+ dp_4^+}{4\pi\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}} \delta(p_1^+ + p_2^+ + p_3^+ + p_4^+) \\
& \qquad \times \Big[a^\dagger(p_1^+) a^\dagger(p_2^+) a^\dagger(p_3^+) a^\dagger(p_4^+) + a(p_1^+) a(p_2^+) a(p_3^+) a(p_4^+)\Big]
\end{aligned}
$$

Graphical $\mathcal{P}^-$ and T

$\mathcal{P}^-$



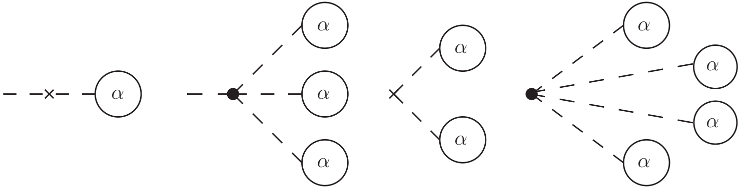
$$T = \alpha \int_0^\infty dp^+ \sqrt{4\pi p^+}\,\Delta(p^+) a^\dagger(p^+) \quad \text{with} \quad \Delta(p^+) \to \delta(p^+)$$



Effective Hamiltonian

For the zero modes

$$\overline{\mathcal{P}^-} = \sqrt{4\pi}\left[\mu^2\alpha + \frac{1}{6}\lambda\alpha^3\right]\int \frac{dp^+}{\sqrt{p^+}}\Delta(p^+)a^\dagger(p^+)$$
$$+4\pi\left[\frac{1}{2}\mu^2\alpha^2 + \frac{1}{24}\lambda\alpha^4\right]\delta(0).$$



Vacuum eigenvalue problem

For a vacuum valence state, the valence eigenvalue problem is

$$P_v\overline{\mathcal{P}^-}|0\rangle = \mathcal{E}^- \int dx^-|0\rangle, \text{ with } \mathcal{E}^- = \frac{1}{2}\mu^2\alpha^2 + \frac{1}{24}\lambda\alpha^4.$$

The auxiliary equation, projected onto the one-zero-mode sector, yields

$$\mu^2\alpha + \frac{1}{6}\lambda\alpha^3 = 0.$$

The solutions are $\alpha = 0$ or $\alpha^2 = -6\mu^2/\lambda$, with α the vev for the field. A coherent-state analysis yields the same results.
If we now consider the wrong-sign case, with $\mu^2 \to -\mu^2$, we find $\alpha = \pm\sqrt{6\lambda}/\mu$, which corresponds to the shift of the field ϕ that brings the Hamiltonian density to a minimum. Thus, the inclusion of a zero mode in the LFCC T operator allows for the necessary shift in the field.

Symmetry breaking

The effective Hamiltonian will have terms that mix Fock states with odd and even numbers of particles, which is characteristic of broken symmetry.

For example, a commutator that contributes to the Baker–Hausdorff expansion of $\overline{\mathcal{P}^-}$ is

$$[\mathcal{P}^-_{\rm int}, T] = \frac{\lambda\alpha}{2}\int \frac{dp_1^+ dp_2^+ dp^+}{\sqrt{4\pi p_1^+ p_2^+ p^+}}\delta(p^+ - p_1^+ - p_2^+) \times \left[a^\dagger(p^+)a(p_1^+)a(p_2^+) + a^\dagger(p_1^+)a^\dagger(p_2^+)a(p^+)\right],$$

which changes particle number by one.

Summary

- in light-front quantization, the vacuum is trivial without zero modes.
- the LFCC method, which solves the LF Hamiltonian eigenvalue problem nonperturbatively, can be extended to include zero modes.
- in simple theories, we have shown that this provides for the expected vev and is consistent with a variational coherent-state analysis.
- a four-zero-mode calculation in ϕ^4 theory is underway, to compute the critical coupling for dynamical symmetry breaking.
- another accessible application is a nonperturbative calculation of the Higgs mechanism.