Solution of the Leading Order Evolution Equation for Generalized Parton Distributions

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Workshop on Hard Processes of the GRK 841

Outline of the talk

Literature

GPDs introduced in

- Müller er al, *Fortsch. Phys.* **42**, 1994
- A. V. Radyushkin, *Phys. Lett.* **B 380**, 1996
- X. D. Ji, *Phys. Rev. Lett.* **78**, 1997

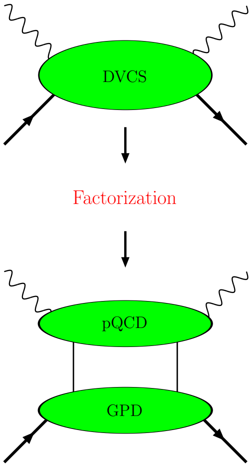
Review articles

- Goeke et al, *Prog. Part. Nucl. Phys.* **47**, 2001
- M. Diehl, *Phys. Rept.* **388**, 2003
- A. V. Belitsky, A. V. Radyushkin, hep-ph/0504030

This work

- A. N. Manashov, M. Kirch, A. Schäfer, *Phys. Rev. Lett.* **95**
- M. Kirch, A. N. Manashov, A. Schäfer, hep-ph/0509330

In a nutshell I



Probing microscopic structure of hadrons:
Hard exclusive processes (DVCS)

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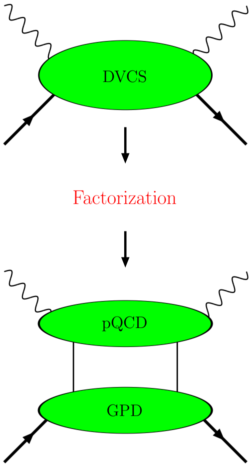

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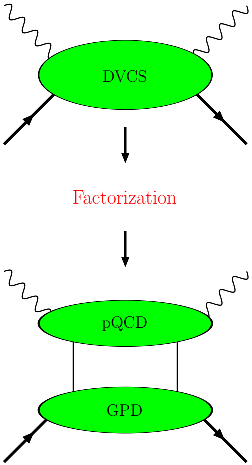

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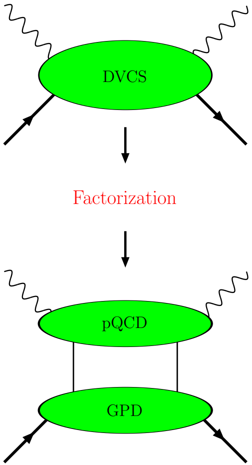

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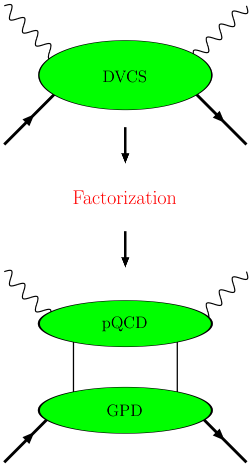

Probing microscopic structure of hadrons:
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Divide Amplitude in

- Process dependend part: pQCD
- Universal nonperturbative part: GPD

In a Nutshell II

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and compare with model calculations

- Predictive power depends heavily on quality of Q^2-evolution!

Matrix elements of composite operators

- GPD: nondiagonal hadronic matrix element

$$\varphi(z_1, z_2) = \langle h(p')|\mathcal{O}(z_1, z_2)|h(p)\rangle$$

of bilocal quark-gluon operator $\mathcal{O}(z_1, z_2)$.

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- Fourier transformed to momentum space

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- Renormalization prescription for composite operator
$\rightarrow$ Scale dependence of GPD (Evolution)

$$\varphi_\xi(z_1, z_2) \equiv \varphi_\xi(z_1, z_2; \mu)$$

Evolution equation and strategy for the solution

- Scale dependence governed by evolution equation (here: in coordinate space)

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \varphi_\xi(z_1, z_2; \mu) = [\mathcal{H}\, \varphi_\xi]\,(z_1, z_2; \mu)$$

$\mathcal{H}$ (Hamiltonian): Integral operator acting on z_1, z_2

- contains counterterms to remove divergencies
- Here: 1-loop $\rightarrow$ conterterms are tree level
- pQCD

[Balitsky, Braun, *Nucl. Phys.* **B311**, 1989]

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- Solution of integro-differential equation:

Expand $\varphi_\xi(z_1, z_2)$ in eigenfunctions of $\mathcal{H}$

Evolution equation and strategy for the solution

- The evolution equation

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$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \varphi(z_1, z_2; \mu) = \alpha_s(\mu)\, \gamma\, \varphi(z_1, z_2; \mu)$$

γ anomalous dimension

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- Can be easiliy integrated

$$\varphi(z_1, z_2; \mu_2) = \left(\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)}\right)^{-\gamma} \varphi(z_1, z_2; \mu_1)$$

Symmetry and traditional approaches

- Tool to find correct set of eigenfunctions: $SL(2,\mathbb{R})$ symmetry of $\mathcal{H}$
- Remnant of conformal symmetry of classical QCD
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- Traditional approach: Use simple scale dependence of Gegenbauer moments of GPD
- But: properties of Gegenbauer polynomials $\Rightarrow$ reconstruction of GPDs from Gegenbauer moments nontrivial
- Many procedures suggested

  [Belitsky, Müller '98; Shuvaev '99; Kivel, Mankiewicz '99;...]

  But have some problems
  - Treat one of the kinematical regions ($|x| < |\xi|$, $|x| > |\xi|$) with help of symmetry and the other by means of analytical continuation
  - numerically unstable or cumbersome

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- Diagonalize $\mathbf{J}^2$ Eigenvalue problem

$$\mathbf{J}^2 \Psi(z_1, z_2) = j(j-1)\Psi(z_1, z_2)$$

has solutions with discrete spectrum, $e^{-i\xi(z_1+z_2)}\Psi_j(z_1 - z_2)$

and continuous, spectrum, $e^{-i\xi(z_1+z_2)}\Psi_j^{\pm}(z_1 - z_2)$

Roadmap to solution of LO-evolution equation II

- Ψ_j and $\Psi_j^{\pm}$ form basis in target space of $\mathbf{J}^2$
- can expand any function from that space over this basis

$$\varphi_\xi(z) \sim \sum_j a(j)\Psi_j(z) + \int \mathrm{d}j\, a^{\pm}(j)\Psi_j^{\pm}(z)$$

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$$\varphi_\xi(z, \mu_2) \sim \sum_j a_{\mu_1}(j) L^{-\gamma}\Psi_j(z) + \int \mathrm{d}j\, a^{\pm}_{\mu_1}(j) L^{-\gamma}\Psi_j(z \pm i\epsilon)$$

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Solution of the evolution equation

$$\varphi_\xi^{\mu_2}(x) = \sum_{j=2}^{\infty} c_\xi^{\mu_1}(j) L^{-\gamma(j)} p_j\left(\frac{x}{\xi}\right) \pm \int_C \frac{\mathrm{d}j}{i\pi} c_\xi^{\mu_1,\pm}(j) L^{-\gamma(j)} q_j\left(\pm\frac{x}{\xi}\right)$$

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- $L = \alpha_s(\mu_1)/\alpha_s(\mu_2)$
- $\gamma(j)$ - anomalous dimension
- $c(j)$ - Expansion coefficients (Input GPD)
- $p_j(x)$, $q_j(x)$ - in terms of Legendre functions

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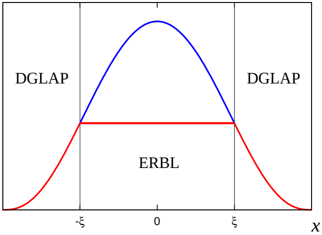

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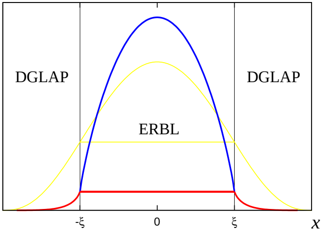

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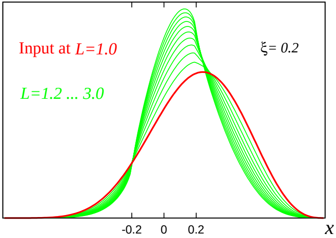
Different properties, limits/asymptotics can be studied.

- Behaviour at $x = \pm\xi$
- large L
- small ξ, x fixed $\rightarrow$ standart DGLAP
- small ξ, x/ξ fixed $\rightarrow$ double scaling

Properties of the solution II

Provides a fast and stable algorithm for numerical evolution!



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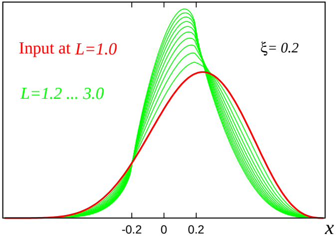

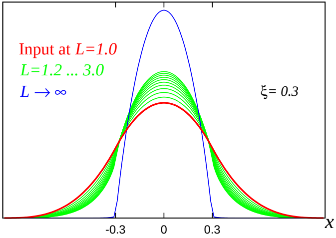

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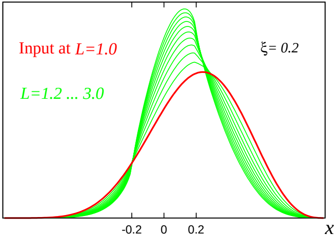

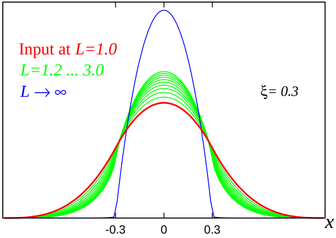

- Algorithm tested on various examples. Shows expected features: Migration of partons, asymptotic behaviour...
- Works also good for small $x \sim 10^{-2} \cdots 10^{-5}$

Summary

- A novel approach to the solution of the LO-evolution equation for twist-2 GPDs has been presented
- Explicit solutions given for non-singlet quark and singlet quark-gluon operators
- The form of the solution is completely determined by the collinear conformal symmetry
- Allows analytical and stable numerical studies
- Further: include polarized GPDs, higher twists, develop efficient code, etc.

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THANK YOU FOR PAYING ATTENTION!