Solution of the Leading Order Evolution Equation for Generalized Parton Distributions

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Workshop on Hard Processes of the GRK 841
Outline of the talk

1. Introduction
   - References and Credits
   - Generalized Parton Distributions
   - Scale dependence of GPDs

2. Evolution equation
   - Strategy
   - Collinear Conformal Symmetry
   - Solution
   - Properties

3. Summary
**Literature**

### GPDs introduced in


### Review articles


### This work

In a nutshell

Probing microscopic structure of hadrons: Hard exclusive processes (DVCS)

Factorization
In a nutshell I

Probing microscopic structure of hadrons:
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Divide Amplitude in

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Divide Amplitude in

- Process dependend part: pQCD
In a nutshell I

Probing microscopic structure of hadrons:
Hard exclusive processes (DVCS)

Factorization

Divide Amplitude in
- Process dependend part: pQCD
- Universal nonperturbative part: GPD
GPD: Information about quark-gluon content of hadrons
**In a Nutshell II**

- GPD: Information about quark-gluon content of hadrons
- Not calculable from first principles of QCD!
GPD: Information about quark- gluon content of hadrons
Not calculable from first principles of QCD!

Extract GPDs from
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Extract GPDs from
Experiments
  technically very involved
  measure GPDs only convoluted with hard scattering kernel
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- Experiments
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- Lattice
  - measures only moments of GPDs (also integrated)
  - not to speak about physical quark masses
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and compare with model calculations

Predictive power depends heavily on quality of $Q^2$-evolution!
Matrix elements of composite operators

- GPD: nondiagonal hadronic matrix element

\[ \varphi(z_1, z_2) = \langle h(p')|O(z_1, z_2)|h(p)\rangle \]

of bilocal quark-gluon operator \( O(z_1, z_2) \).
Matrix elements of composite operators

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of bilocal quark-gluon operator \( \mathcal{O}(z_1, z_2) \).

Fourier transformed to momentum space

\[ \varphi_\xi(z_1, z_2) \sim e^{-i \xi (z_1 + z_2)} \int dx \ e^{ix(z_1 - z_2)} \varphi(x, \xi) \]
Matrix elements of composite operators

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- Renormalization prescription for composite operator

\[ \rightarrow \text{Scale dependence of GPD (Evolution)} \]

\[ \varphi_\xi(z_1, z_2) \equiv \varphi_\xi(z_1, z_2; \mu) \]
Scale dependence governed by evolution equation (here: in coordinate space)

\[ \mu \frac{d}{d\mu} \varphi_\xi (z_1, z_2; \mu) = [\mathcal{H} \varphi_\xi ] (z_1, z_2; \mu) \]

\( \mathcal{H} \) (Hamiltonian): Integral operator acting on \( z_1, z_2 \)
- contains counterterms to remove divergencies
- Here: 1-loop \( \rightarrow \) counterterms are tree level
- pQCD

Evolution equation and strategy for the solution

- Scale dependence governed by evolution equation (here: in coordinate space)

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- contains counterterms to remove divergencies
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- Solution of integro-differential equation:
  Expand \( \varphi_\xi(z_1, z_2) \) in eigenfunctions of \( \mathcal{H} \)
The evolution equation

\[ \mu \frac{d}{d\mu} \varphi(z_1, z_2; \mu) = [\mathcal{H} \varphi](z_1, z_2; \mu) \]
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If \( \varphi(z_1, z_2) \) are eigenfunctions of \( \mathcal{H} \)
Evolution equation and strategy for the solution

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- If \( \varphi(z_1, z_2) \) are eigenfunctions of \( \mathcal{H} \)

\[ \mu \frac{d}{d\mu} \varphi(z_1, z_2; \mu) = \alpha_s(\mu) \gamma \varphi(z_1, z_2; \mu) \]

\( \gamma \) anomalous dimension
Evolution equation and strategy for the solution

- The evolution equation

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- If \( \varphi(z_1, z_2) \) are eigenfunctions of \( \mathcal{H} \)

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\( \gamma \) anomalous dimension

- Can be easily integrated

\[ \varphi(z_1, z_2; \mu_2) = \left( \frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} \right)^{-\gamma} \varphi(z_1, z_2; \mu_1) \]
Symmetry and traditional approaches

- Tool to find correct set of eigenfunctions: $SL(2, \mathbb{R})$ symmetry of $\mathcal{H}$
- Remnant of conformal symmetry of classical QCD survives in LO counterterms
Symmetry and traditional approaches

- Tool to find correct set of eigenfunctions: $SL(2, \mathbb{R})$ symmetry of $H$
- Remnant of conformal symmetry of classical QCD
  \[ \rightarrow \text{survives in LO counterterms} \]

- Traditional approach: Use simple scale dependence of Gegenbauer moments of GPD
- But: properties of Gegenbauer polynomials \[ \Rightarrow \text{reconstruction of GPDs from Gegenbauer moments nontrivial} \]
- Many procedures suggested
  
  [Belitsky, Müller '98; Shuvaev '99; Kivel, Mankiewicz '99;...]
  
  But have some problems
  - Treat one of the kinematical regions ($|x| < |\xi|$, $|x| > |\xi|$) with help of symmetry and the other by means of analytical continuation
  - numerically unstable or cumbersome
Start with GPD $\varphi_\xi(z_1, z_2)$ in coordinate space
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Exploit rigorously collinear conformal symmetry
- Find "suitable" representation of $SL(2, \mathbb{R})$ under which GPD transforms.
- Determine Casimir operator $J^2$.
- Find $[H, J^2] = 0$
Roadmap to solution of LO-evolution equation I

- Start with GPD $\varphi_\xi(z_1, z_2)$ in coordinate space

- Exploit **rigorously** collinear conformal symmetry
  - Find "suitable" representation of $SL(2, \mathbb{R})$ under which GPD transforms.
  - Determine Casimir operator $J^2$.
  - Find $[\mathcal{H}, J^2] = 0$

- Diagonalize $J^2$ Eigenvalue problem

\[ J^2 \psi(z_1, z_2) = j(j - 1)\psi(z_1, z_2) \]

has solutions with **discrete** spectrum, $e^{-i\xi(z_1+z_2)}\psi_j(z_1 - z_2)$

and continuous, spectrum, $e^{-i\xi(z_1+z_2)}\psi_j^\pm(z_1 - z_2)$
\( \psi_j \) and \( \psi_j^{\pm} \) form basis in target space of \( \mathbf{J}^2 \)

- can expand any function from that space over this basis

\[
\varphi_\xi(z) \sim \sum_j a(j) \psi_j(z) + \int \, \text{d}j \, a^{\pm}(j) \psi_j^{\pm}(z)
\]
\(\psi_j\) and \(\psi_j^{\pm}\) form basis in target space of \(J^2\)

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- \(\psi_j\) eigenfunction of \(\mathcal{H}\) – BUT \(\psi_j^{\pm}\) NOT! (\(\mathcal{H}\) non-hermitean)
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\( \psi_j \) eigenfunction of \( \mathcal{H} \) – BUT \( \psi_j^\pm \) NOT! (\( \mathcal{H} \) non-hermitean)

We are lucky: Can rewrite expansion only in terms of \( \psi_j(z) \)

\[
\varphi_\xi(z) \sim \sum_j a(j) \psi_j(z) + \int \, dj \ a^\pm(j) \psi_j(z \pm i\epsilon)
\]
ψ_j and ψ_j± form basis in target space of J^2

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\[ \varphi_\xi(z) \sim \sum_j a(j)\psi_j(z) + \int dj a^\pm(j)\psi^\pm_j(z) \]

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\[ \varphi_\xi(z) \sim \sum_j a(j)\psi_j(z) + \int dj a^\pm(j)\psi_j(z \pm i\epsilon) \]

- Insert expansion into evolution equation
ψj and ψj± form basis in target space of J^2

can expand any function from that space over this basis

\[ \varphi_\xi(z) \sim \sum_j a(j) \psi_j(z) + \int dj \ a^{\pm}(j) \psi^{\pm}_j(z) \]

ψj eigenfunction of \( H \) – BUT ψj± NOT! (\( H \) non-hermitean)

We are lucky: Can rewrite evolution only in terms of \( \psi_j^{\mu_2}(z) \)

\[ \varphi_\xi(z, \mu_2) \sim \sum_j a^{\mu_1}(j) L^{-\gamma} \psi_j(z) + \int dj \ a^{\pm}_{\mu_1}(j) L^{-\gamma} \psi_j(z \pm i\epsilon) \]

Insert expansion into evolution equation
ψ_± and ψ_± form basis in target space of J^2

- can expand any function from that space over this basis

\[ \varphi_\xi(z) \sim \sum_j a(j) \psi_j(z) + \int dj \ a^\pm(j) \psi^\pm_j(z) \]

- ψ_j eigenfunction of \( \mathcal{H} \) – BUT ψ_± NOT! (\( \mathcal{H} \) non-hermitean)

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\[ \varphi_\xi(z, \mu_2) \sim \sum_j a_{\mu_1}(j) L^{-\gamma} \psi_j(z) + \int dj \ a^\pm_{\mu_1}(j) L^{-\gamma} \psi_j(z \pm i\epsilon) \]

- Insert expansion into evolution equation
- Fourier transform to momentum space
Solution of the evolution equation

\[ \varphi_{\xi}^{\mu}(x) = \sum_{j=2}^{\infty} c_{\xi}^{\mu}(j) L^{-\gamma(j)} p_j \left( \frac{x}{\xi} \right) \pm \int_{C} \frac{dj}{i\pi} c_{\xi}^{\mu,\pm}(j) L^{-\gamma(j)} q_j \left( \pm \frac{x}{\xi} \right) \]

Completely fixed by group (representation) theory!
Solution of the evolution equation

\[ \varphi_{\xi}^{\mu_2}(x) = \sum_{j=2}^{\infty} c_{\xi}^{\mu_1}(j) L^{-\gamma(j)} p_j \left( \frac{x}{\xi} \right) \pm \int \frac{dj}{i\pi} c_{\xi}^{\mu_1,\pm}(j) L^{-\gamma(j)} q_j \left( \pm \frac{x}{\xi} \right) \]

Completely fixed by group (representation) theory!

- \( L = \alpha_s(\mu_1)/\alpha_s(\mu_2) \)
- \( \gamma(j) \) - anomalous dimension
- \( c(j) \) - Expansion coefficients (Input GPD)
- \( p_j(x), q_j(x) \) - in terms of Legendre functions
Solution of the evolution equation

\[ \varphi_{\xi}^{\mu}(x) = \sum_{j=2}^{\infty} c_{\xi}^{\mu}(j)p_{j}\left(\frac{x}{\xi}\right) \pm \int_{\mathcal{C}} \frac{dj}{i\pi} c_{\xi}^{\mu,\pm}(j)q_{j}\left(\pm\frac{x}{\xi}\right) \]

Completely fixed by group (representation) theory!

- \( L = 1 \)
- \( \gamma(j) \) - anomalous dimension
- \( c(j) \) - Expansion coefficients (Input GPD)
- \( p_{j}(x), q_{j}(x) \) - in terms of Legendre functions
Solution of the evolution equation

\[ \varphi^\mu_\xi(x) = \sum_{j=2}^{\infty} c^\mu_\xi(j) L^{-\gamma(j)} p_j \left( \frac{x}{\xi} \right) \pm \int_C \frac{dj}{i\pi} c^\mu_\xi(\pm j) L^{-\gamma(j)} q_j \left( \pm \frac{x}{\xi} \right) \]

Completely fixed by group (representation) theory!

- \( L > 1 \)
- \( \gamma(j) \) - anomalous dimension
- \( c(j) \) - Expansion coefficients (Input GPD)
- \( p_j(x), q_j(x) \) - in terms of Legendre functions
Properties of the solution

The solution can be used both for analytical and numerical studies of the evolution!
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The solution can be used both for **analytical** and **numerical** studies of the evolution!

Solution has been presented for twist-2 non-singlet quark and singlet quark-gluon GPDs (including mixing).
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Solution has been presented for twist-2 non-singlet quark and singlet quark- gluon GPDs (including mixing).

Different properties, limits/asymptotics can be studied.

- Behaviour at $x = \pm \xi$
- large $L$
- small $\xi$, $x$ fixed $\rightarrow$ standart DGLAP
- small $\xi$, $x/\xi$ fixed $\rightarrow$ double scaling
Properties of the solution II

Provides a **fast and stable** algorithm for numerical evolution!
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![Graphs showing input at different values of L for ξ = 0.2 and ξ = 0.3](image-url)
Provides a **fast and stable** algorithm for numerical evolution!

- Algorithm tested on various examples. Shows expected features: Migration of partons, asymptotic behaviour...
- Works also good for small $x \sim 10^{-2} \ldots 10^{-5}$
A novel approach to the solution of the LO-evolution equation for twist-2 GPDs has been presented

- Explicit solutions given for non-singlet quark and singlet quark-gluon operators
- The form of the solution is completely determined by the collinear conformal symmetry
- Allows analytical and stable numerical studies
- Further: include polarized GPDs, higher twists, develop efficient code, etc.
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THANK YOU FOR PAYING ATTENTION!