Recent Developments in QCD II

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How to calculate scattering amplitudes

- Off-shell methods
  Traditional Feynman diagram approach for off-shell Greens functions
- On-shell methods
  Based on S-matrix ideas of 1960's but recently inspired by Witten's proposal to relate perturbative gauge theory amplitudes to topological string theory in twistor space

  Witten, hep-th/0312171

  $\Rightarrow$ MHV rules

  $\Rightarrow$ BCFW recursion relations

  $\Rightarrow$ Generalised unitarity
- Common methods
  - Colour ordered amplitudes
  - Spinor helicity approach

Example

Multi-jet production at the LHC using HELAC/PHEGAS

Draggiotis, Kleiss, Papadopoloulos

# of jets	2	3	4	5	6	7	8
# of dist.processes	10	14	28	36	64	78	130
total # of processes	126	206	621	861	1862	2326	4342
$\sigma(nb)$	-	91.41	6.54	0.458	0.030	0.0022	0.00021
% Gluonic	-	45.7	39.2	35.7	35.1	33.8	26.6

- The number of Feynman diagrams for an n gluon process increases very quickly with n

$\Rightarrow$ for the 10 gluon amplitude there are 10,525,900 diagrams

$\Rightarrow$ Feynman diagrams very inefficient for many legs

- Control the quantum numbers of the scattering particles

Multiparticle production

The number of tree Feynman diagrams for an n gluon process increases very quickly with n

n	diagrams
4	4
5	25
6	220
7	2485
8	34300
9	559405
10	10525900

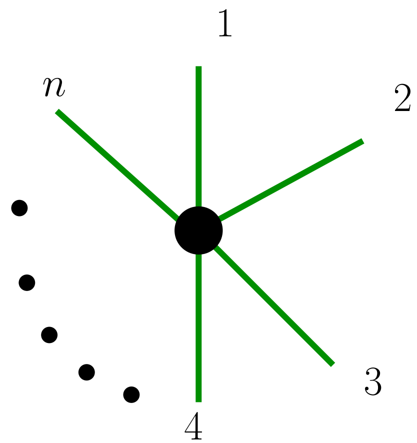
$\Rightarrow$ Feynman diagram evaluation is very inefficient for many legs
- too many diagrams, terms per diagram, kinematic variables

Colour Ordered Amplitudes

$$\mathcal{A}_n(1,\ldots,n) = \sum_{perms} Tr(T^{a_1}\ldots T^{a_n})A_n(1,\ldots,n)$$

Colour-stripped amplitudes A_n: cyclically ordered permutations

Order of external gluons fixed

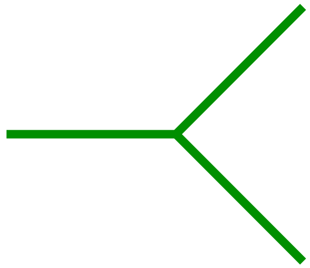


The subamplitudes A_n are
(a) gauge invariant
(b) have nice properties in the infrared limits.

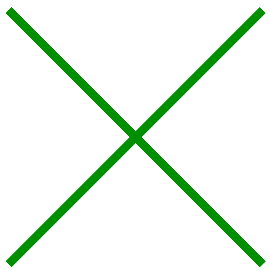
Can reconstruct the full amplitude $\mathcal{A}_n$ from A_n.
In the large N limit,

$$|\mathcal{A}_n(1,\ldots,n)|^2 \sim N^{n-2}(N^2-1)\sum_{perms}|A_n(1,\ldots,n)|^2$$

Colour Ordered Feynman Rules



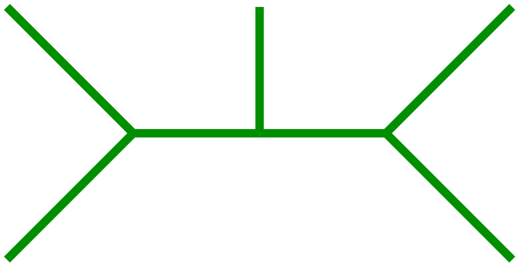
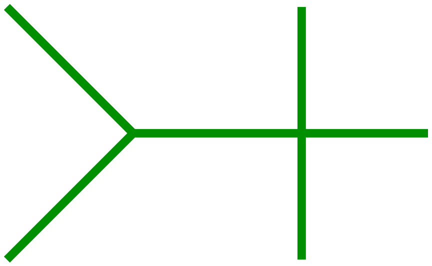
$$i\left(\left(p_1^{\mu_3}-p_2^{\mu_3}\right)g^{\mu_1\mu_2}+\left(p_2^{\mu_1}-p_3^{\mu_1}\right)g^{\mu_2\mu_3}+\left(p_3^{\mu_2}-p_1^{\mu_2}\right)g^{\mu_3\mu_1}\right)$$



$$-i\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4}+g^{\mu_1\mu_4}g^{\mu_3\mu_2}-2g^{\mu_1\mu_3}g^{\mu_2\mu_4}\right)$$

Only calculate diagrams with cyclic colour ordering

Example:

$A_5 =$  $+$ 

i.e. 10 diagrams rather than 25

Power of colour ordering

n	diagrams	colour ordered diagrams
4	4	3
5	25	10
6	220	36
7	2485	133
8	34300	501
9	559405	1991
10	10525900	7335

$\Rightarrow$ Big reduction in number of diagrams

but still too many diagrams

Spinor Helicity Formalism

- Spinor for a massless fermion, momentum p:

$$\not{p}\,u(p) = 0, \qquad\qquad |p\pm\rangle = u_\pm(p) = \frac{1}{2}\left(1 \pm \gamma_5\right) u(p)$$

- Spinor products:

$$\begin{aligned} \langle ij\rangle &= \langle p_i - |p_j+\rangle = \overline{u_-(p_i)}u_+(p_j) \\ [ij] &= \langle p_i + |p_j-\rangle = \overline{u_+(p_i)}u_-(p_j) \end{aligned}$$

- Spinor products are complex numbers and have numerical representations
- Dot products

$$s_{ij} = (p_i + p_j)^2 = 2\,p_i \cdot p_j = \langle ij\rangle[ji]$$

Spinor Helicity Formalism

- Polarisation vector for a massless gauge boson, momentum p:

$$\epsilon_\mu^\pm(p,\eta) = \pm \frac{\langle p \pm |\gamma_\mu|\eta \pm\rangle}{\sqrt{2}\langle \eta \mp |p \pm\rangle}$$

- Easy to show that:

$$\epsilon^\pm \cdot \epsilon^{\pm *} = -1, \qquad p \cdot \epsilon(p,\eta) = 0, \qquad \epsilon^\pm \cdot \epsilon^{\mp *} = 0.$$

- η is a light-like axial gauge vector

$$\sum \epsilon_\mu^\pm(p,\eta)\epsilon_\nu^\pm(p,\eta) = -g_{\mu\nu} + \frac{p_\mu \eta_\nu + p_\nu \eta_\mu}{p \cdot \eta}$$

- amplitudes are η independent
  sensible choice kills many diagrams

Spinor Helicity Formalism

- In Weyl (chiral) representation, each helicity state is represented by a bi-spinor ($a = 1, 2$)

$$u_+(p) = \lambda_{pa}, \qquad u_-(p) = \tilde{\lambda}_p^{\dot{a}},$$
$$\overline{u_+(p)} = \tilde{\lambda}_{p\dot{a}}, \qquad \overline{u_-(p)} = \lambda_p^a$$

so that

$$\begin{aligned} \langle ij \rangle &= \overline{u_-(p_i)} u_+(p_j) = \lambda_i^a \lambda_{ja} = \epsilon_{ab} \lambda_i^a \lambda_j^b \\ [ij] &= \overline{u_+(p_i)} u_-(p_j) = \tilde{\lambda}_{i\dot{a}} \tilde{\lambda}_j^{\dot{a}} = -\epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}} \end{aligned}$$

- We can write massless vector

$$p_{a\dot{a}} \equiv p_\mu \sigma^\mu_{a\dot{a}} = \lambda_{pa} \tilde{\lambda}_{p\dot{a}}$$

Spinor Helicity Formalism

- Polarisation vectors for particle i:

$$\varepsilon^{-}_{ia\dot{a}} = \frac{\lambda_{ia}\tilde{\eta}_{\dot{a}}}{[\lambda_i \eta]}, \qquad \varepsilon^{+}_{ia\dot{a}} = \frac{\eta_a \tilde{\lambda}_{i\dot{a}}}{\langle \eta \lambda_i \rangle}$$

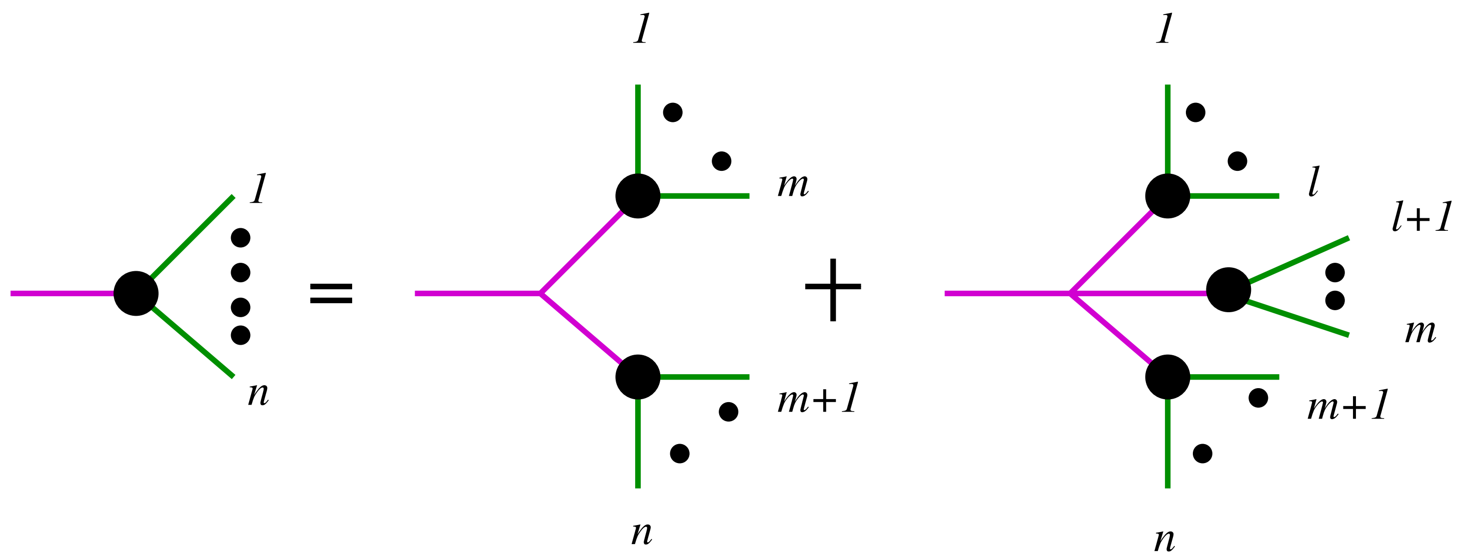
- For real momenta in Minkowski space,

$$\tilde{\lambda} = \lambda^*$$

- For space-time signature $(+,+,-,-)$, $\tilde{\lambda}, \lambda$ are real and independent
- Amplitudes are functions of the λ_i and $\tilde{\lambda}_i$

Recursion relations

Full amplitudes can be built up from simpler amplitudes with fewer particles



Purple gluons are off-shell, green gluons are on-shell.
This is a recursion relation built from off-shell currents.

Berends, Giele

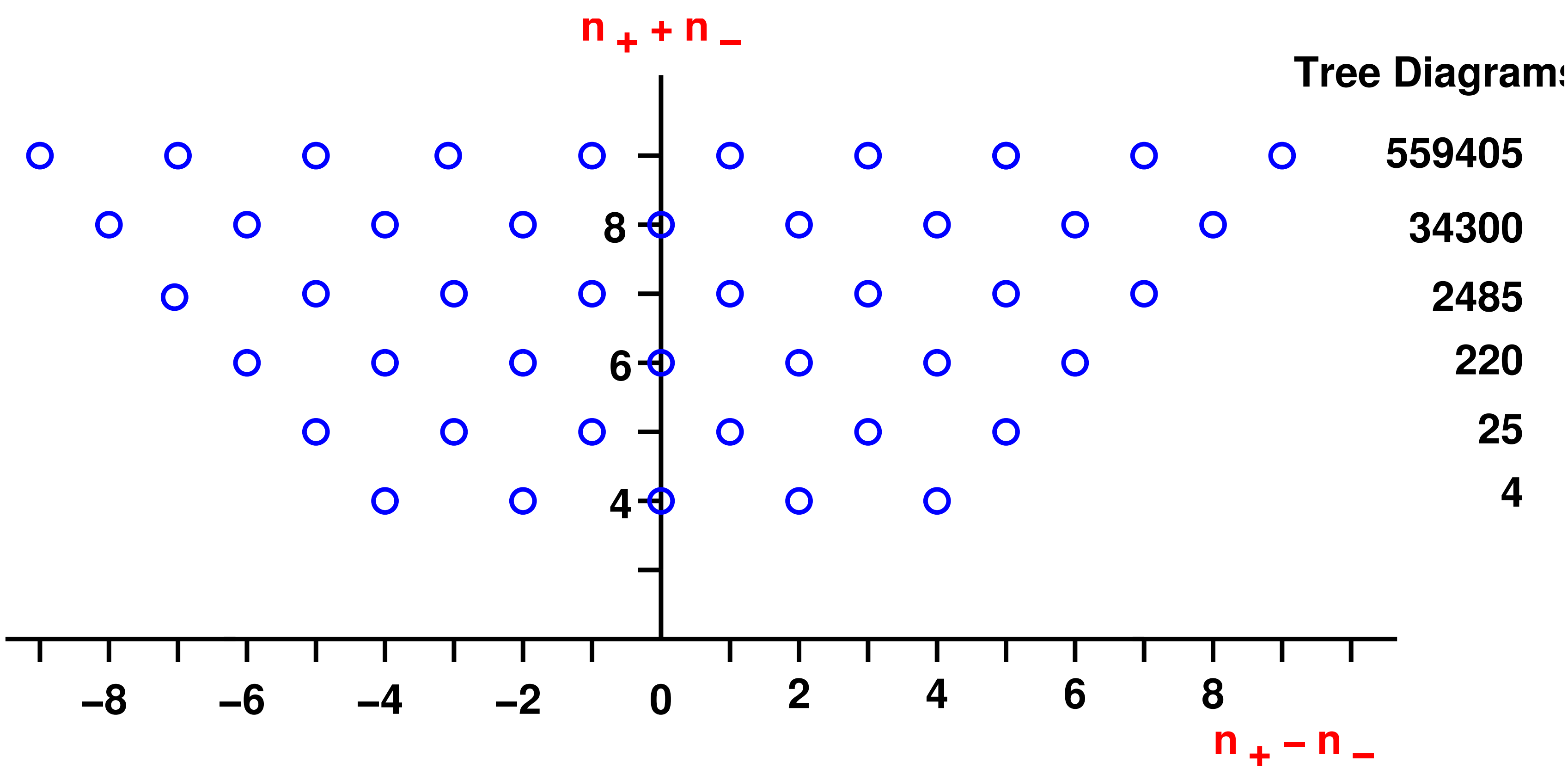
Particularly suited to numerical solution

`ALPGEN`, `HELAC/PHEGAS`

Off-shell methods: Feynman diagrams

- ✓ Direct link to Lagrangian
- ✓ Easy to adapt to any model
- ✓ Easy to include massive particles with/without spin
- ✓ Easy to automate
  $\Rightarrow$ tree-level packages `Madgraph/Grace/CompHep/...`
- ✓ Off-shell Berends-Giele recursion relations
  $\Rightarrow$ tree-level packages `Alpgen/HELAC/PHEGAS/...`
- ✗ Many Feynman diagrams
- ✗ Large cancellations between diagrams
- ✗ Loop amplitudes manpower intensive

Gluonic helicity amplitudes



Each row describes scattering with n_+ positive helicities and n_- negative helicities.
Each circle represents an allowed helicity configuration - from all positive on the right to all negative on the left

Gluonic helicity amplitudes

For example, the result of computing the 10 colour ordered diagrams for the five-gluon process yields

$$
\begin{aligned}
A_5(1^\pm, 2^+, 3^+, 4^+, 5^+) &= 0 \\
A_5(1^-, 2^-, 3^+, 4^+, 5^+) &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}
\end{aligned}
$$

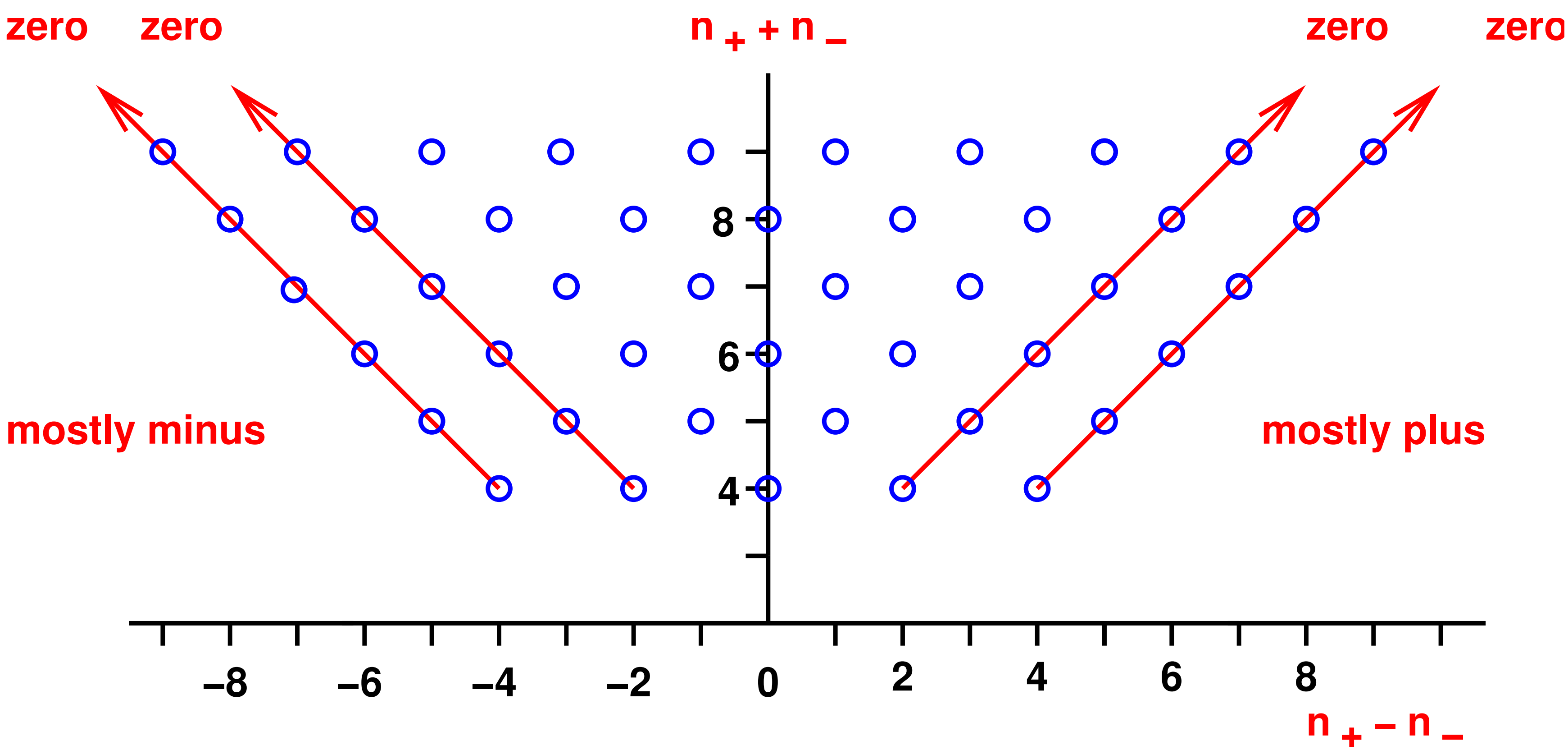
In fact, for n point amplitudes,

$$
\begin{aligned}
A_n(1^\pm, 2^+, 3^+, , \ldots, n^+) &= 0 \\
A_n(1^-, 2^-, 3^+, \ldots, n^+) &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}
\end{aligned}
$$

Maximally helicity violating (MHV) amplitudes

Parke, Taylor; Berends, Giele

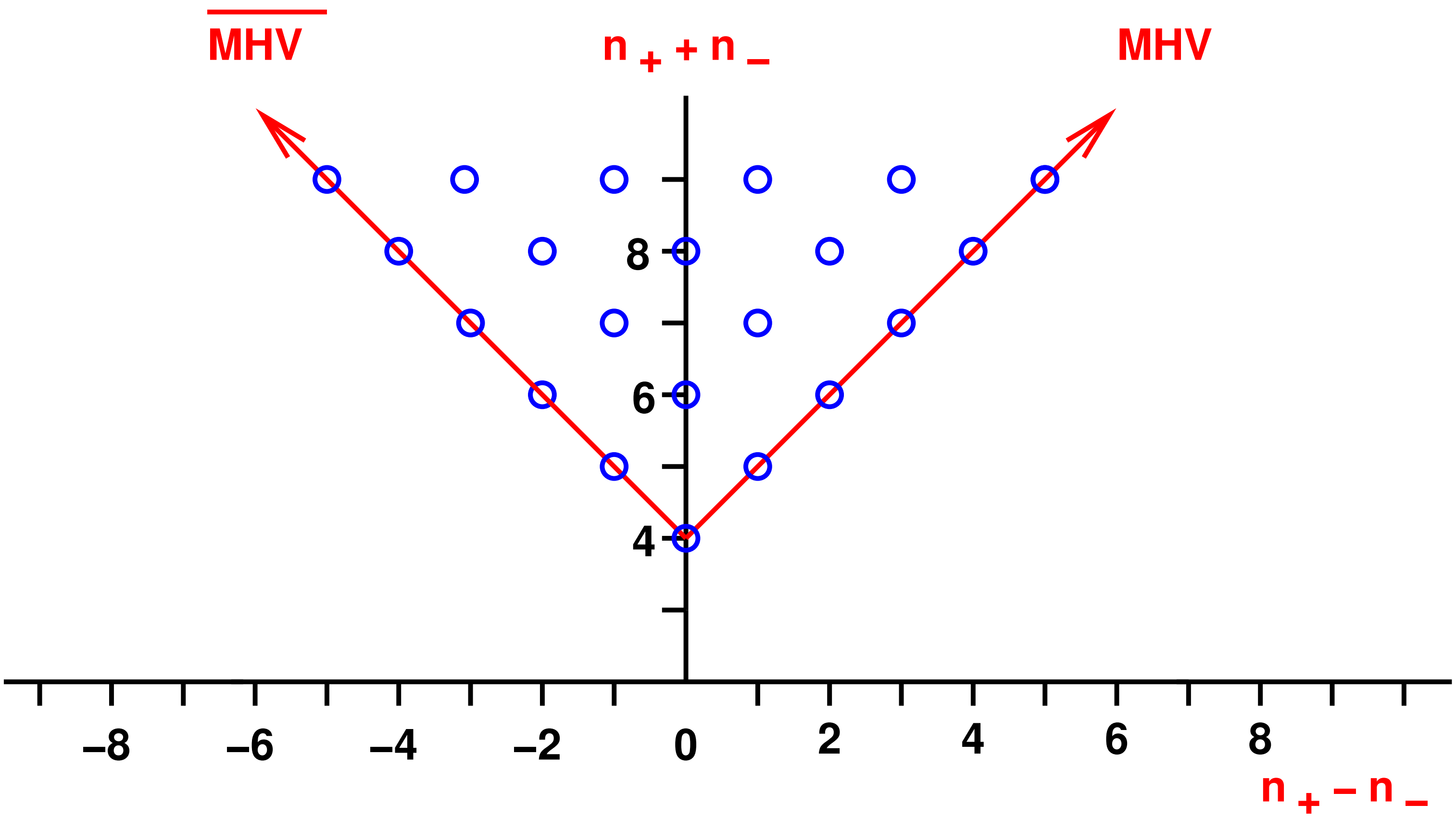
Gluonic helicity amplitudes



$$A_n(1^{\pm}, 2^{+}, 3^{+}, , \ldots, n^{+}) = 0$$

effective tree-level supersymmetry

Gluonic helicity amplitudes



$$A_n(1^-, 2^-, 3^+, \ldots, n^+) \quad = \quad \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Twistor Space

Penrose, 1967

Amplitudes in twistor space obtained by Fourier transform with respect to positive helicity spinors,

$$\tilde{\lambda}_{\dot{a}} = i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad \mu^{\dot{a}} = i\frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}}$$

Momentum conservation yields

$$\delta^4\left(\sum k_i\right) = \int d^4x \exp\left(ix^{a\dot{a}}\sum_j \lambda_{ja}\tilde{\lambda}_{j\dot{a}}\right)$$

so that the amplitude in twistor space is

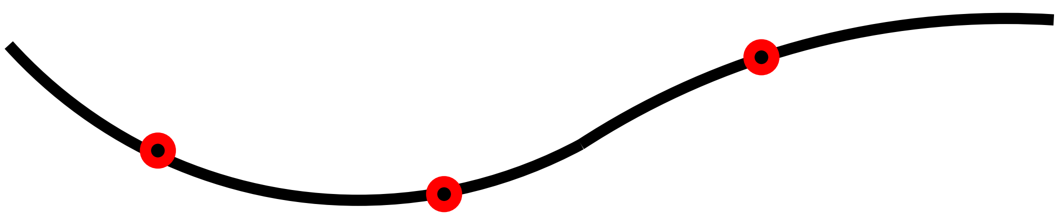
$$\tilde{A}(\lambda_i, \mu_i) = \int d^4x \int \prod_i \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} \exp\left(i\sum_j \left(\mu_j^{\dot{a}} + x^{a\dot{a}}\lambda_{ja}\right)\tilde{\lambda}_{j\dot{a}}\right) A(\lambda_i, \tilde{\lambda}_i)$$

Twistor Space

Witten, hep-th/0312171

Witten observed that in twistor space external points lie on certain algebraic curves
⇒ degree of curve is related to the number of negative helicities and loops

$$d = n_- - 1 + l$$



MHV amplitudes in Twistor Space

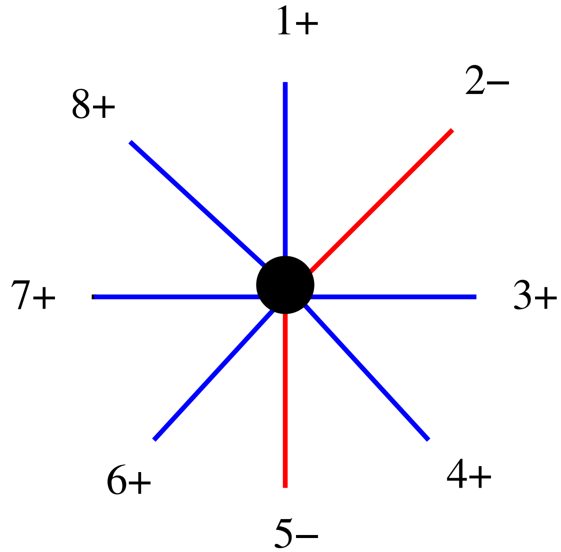
MHV amplitudes depend only on the λ_i variables

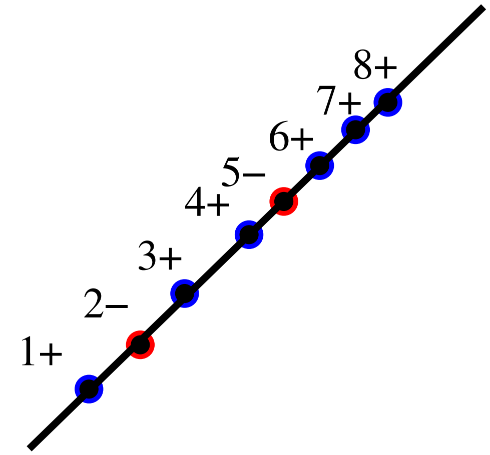
$$\begin{aligned}
\tilde{A}_{MHV}(\lambda_i, \mu_{\dot{i}}) &= \int d^4x \int \prod_i \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} \exp\left(i \sum_j \left(\mu_j^{\dot{a}} + x^{a\dot{a}} \lambda_{ja} \right) \tilde{\lambda}_{j\dot{a}} \right) A(\lambda_i, \tilde{\lambda}_i) \\
&\propto \delta^2 \left(\sum_j \left(\mu_j^{\dot{a}} + x^{a\dot{a}} \lambda_{ja} \right) \right)
\end{aligned}$$

which is only non-zero on a straight line in $(\lambda_i, \mu_{\dot{i}})$ space.

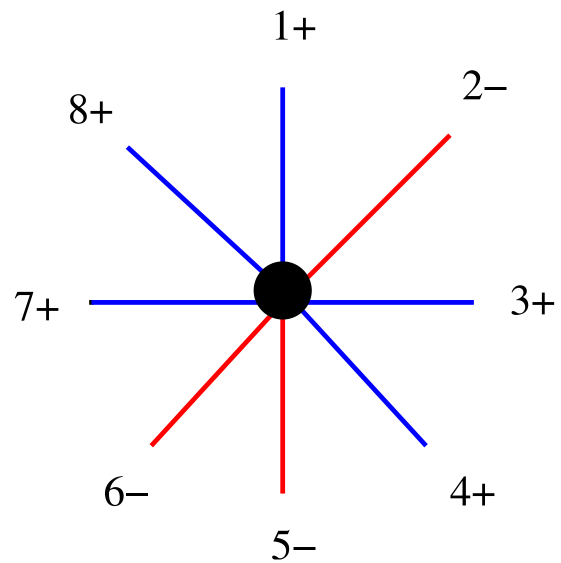
Twistor Space

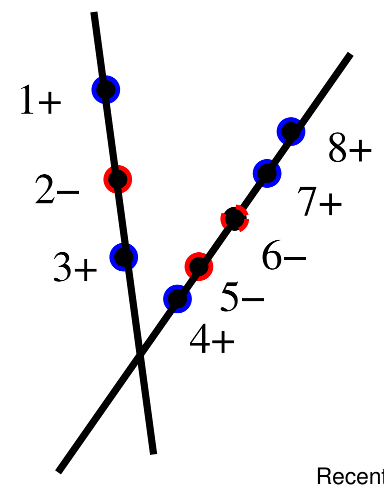
MHV





NMHV



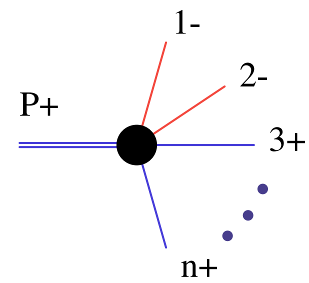


MHV rules

Start from on-shell MHV amplitude and define off-shell vertices

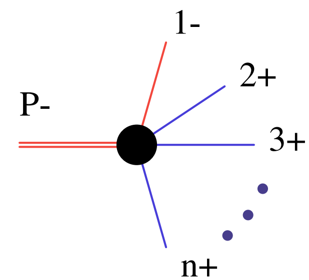
Cachazo, Svrcek and Witten

$$V(1^-,2^-,3^+,\ldots,n^+,P^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\cdots\langle n-1n\rangle\langle nP\rangle\langle P1\rangle}$$



and

$$V(1^-,2^+,3^+,\ldots,n^+,P^-) = \frac{\langle 1P\rangle^4}{\langle 12\rangle\cdots\langle n-1n\rangle\langle nP\rangle\langle P1\rangle}$$



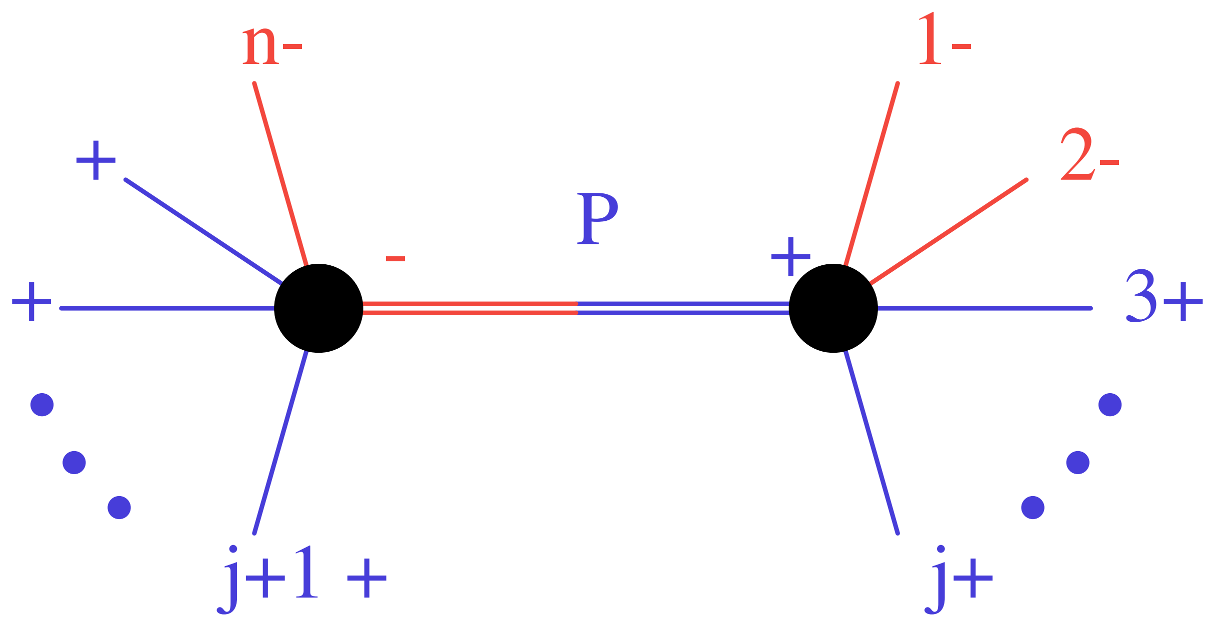
Crucial step is off-shell continuation $P^2 \neq 0$:

$$\langle iP\rangle = \frac{\langle i^-|\not{P}|\eta^-]}{[P\eta]} = \sum_j \frac{\langle i^-|\not{j}|\eta^-]}{[P\eta]}$$

where $P = \sum_j j$ and η is lightlike auxiliary vector

MHV rules

Must connect up a positive helicity off-shell line with a negative helicity off-shell line



Connecting two MHV's $\Rightarrow$ amplitude with 3 negative helicities
Connecting three MHV's $\Rightarrow$ amplitude with 4 negative helicities
etc.

Example: six gluon scattering

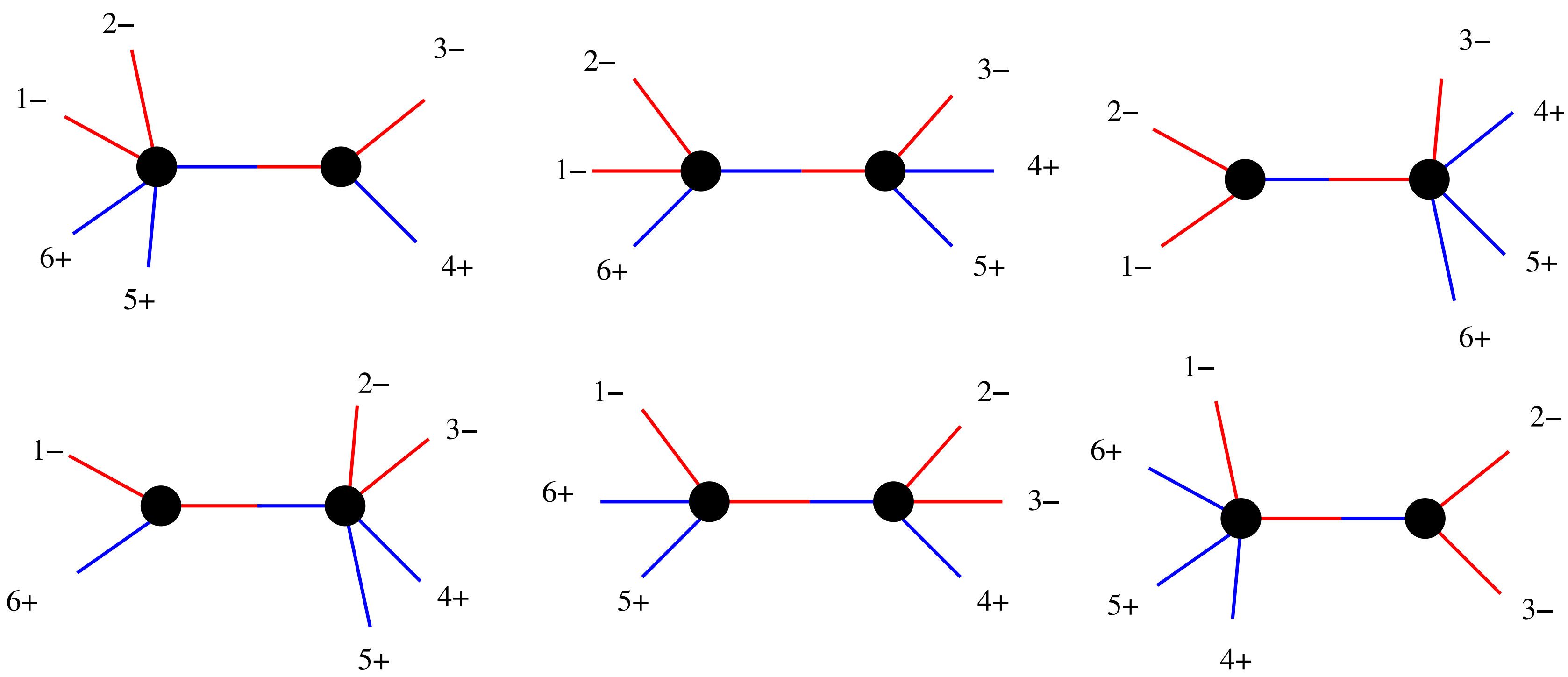
As an example, lets use the MHV rules to calculate one of the first non-MHV amplitudes

$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$$

Step 1 Draw all the allowed MHV diagrams

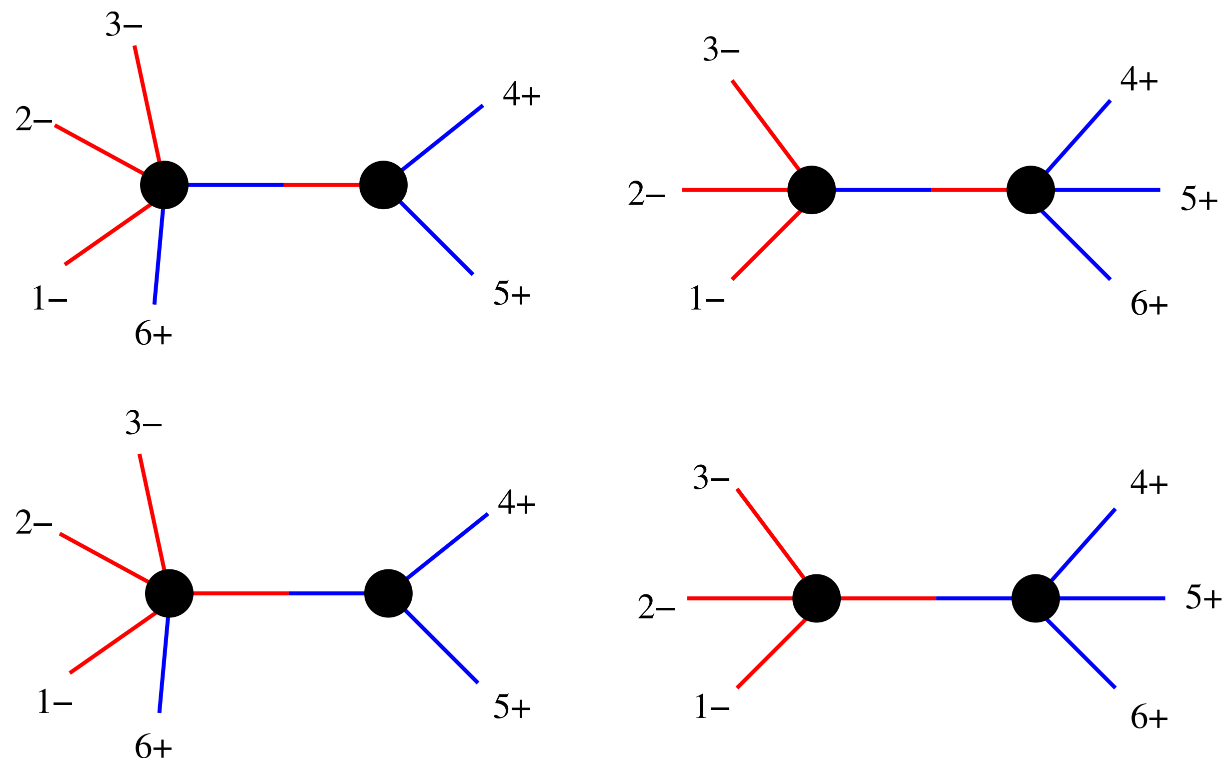
Example: six gluon scattering

There are six MHV graphs



Example: six gluon scattering

Some graphs are not allowed e.g.



Example: six gluon scattering

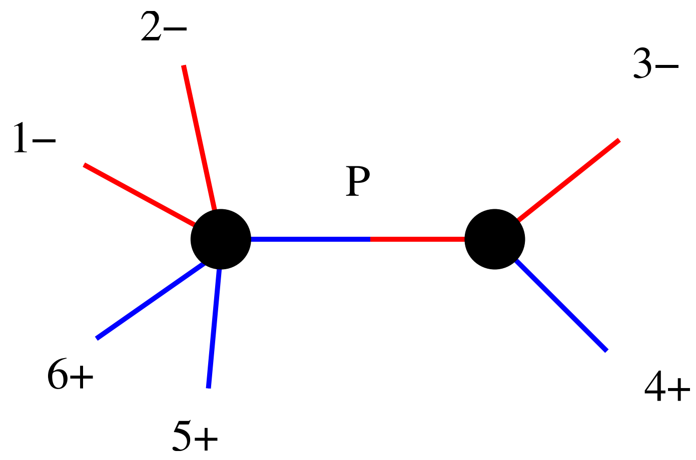
As an example, lets use the MHV rules to calculate one of the first non-MHV amplitudes

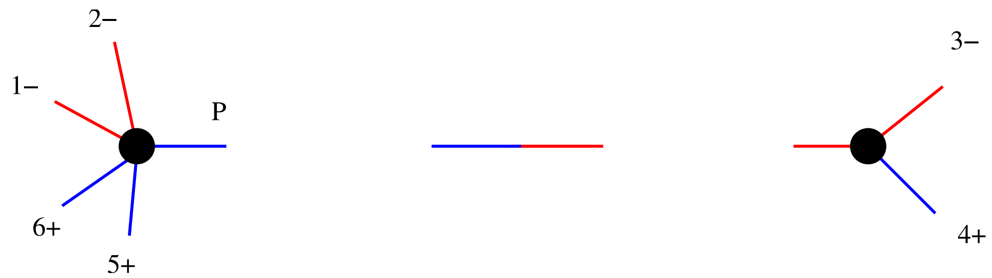
$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$$

Step 1 Draw all the allowed MHV diagrams

Step 2 Apply MHV rules to each diagram

Example: six gluon scattering: diagram 1

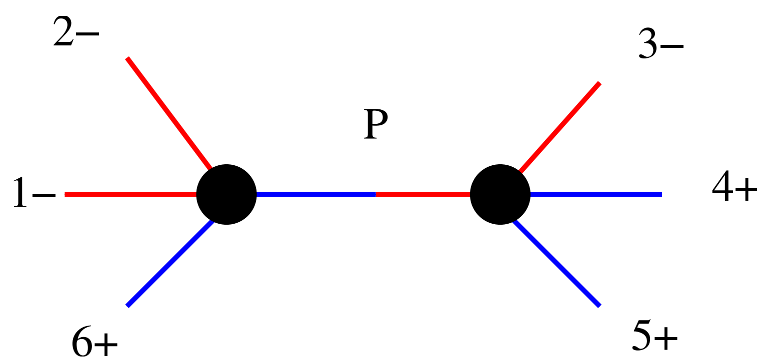


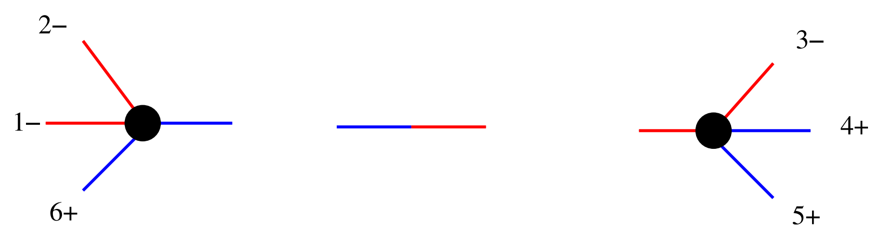


$$\frac{\langle 12\rangle^4}{\langle 56\rangle\langle 61\rangle\langle 12\rangle\langle 2|P|\eta\rangle\langle 5|P|\eta\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|P|\eta\rangle^4}{\langle 34\rangle\langle 4|P|\eta\rangle\langle 3|P|\eta\rangle}$$

with $P = 3 + 4 = -(1 + 2 + 5 + 6)$

Example: six gluon scattering: diagram 2





$$\frac{\langle 12\rangle^4}{\langle 61\rangle\langle 12\rangle\langle 2|P|\eta\rangle\langle 6|P|\eta\rangle} \times \frac{1}{s_{345}} \times \frac{\langle 3|P|\eta\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 5|P|\eta\rangle\langle 3|P|\eta\rangle}$$

with $P = 3 + 4 + 5 = -(1 + 2 + 6)$

Example: six gluon scattering

As an example, lets use the MHV rules to calculate one of the first non-MHV amplitudes

$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$$

Step 1 Draw all the allowed MHV diagrams

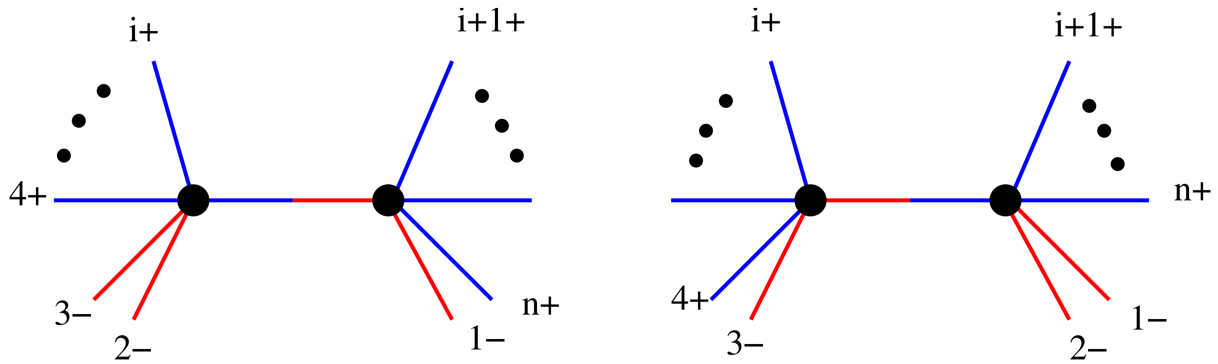
Step 2 Apply MHV rules to each diagram

Step 3 Add up diagrams and check η independence

Next-to MHV amplitude for n gluons

Simplest case: $A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ $\qquad$ $2(n-3)$ graphs

Cachazo, Svrcek and Witten



$$A = \sum_{i=3}^{n-1} \frac{\langle 1(2,i)\rangle^3}{\langle (2,i)i+1\rangle\langle i+1 i+2\rangle \ldots \langle n1\rangle} \frac{1}{s_{2,i}^2} \frac{\langle 23\rangle^3}{\langle (2,i)2\rangle\langle 34\rangle \cdots \langle i(2,i)\rangle}$$

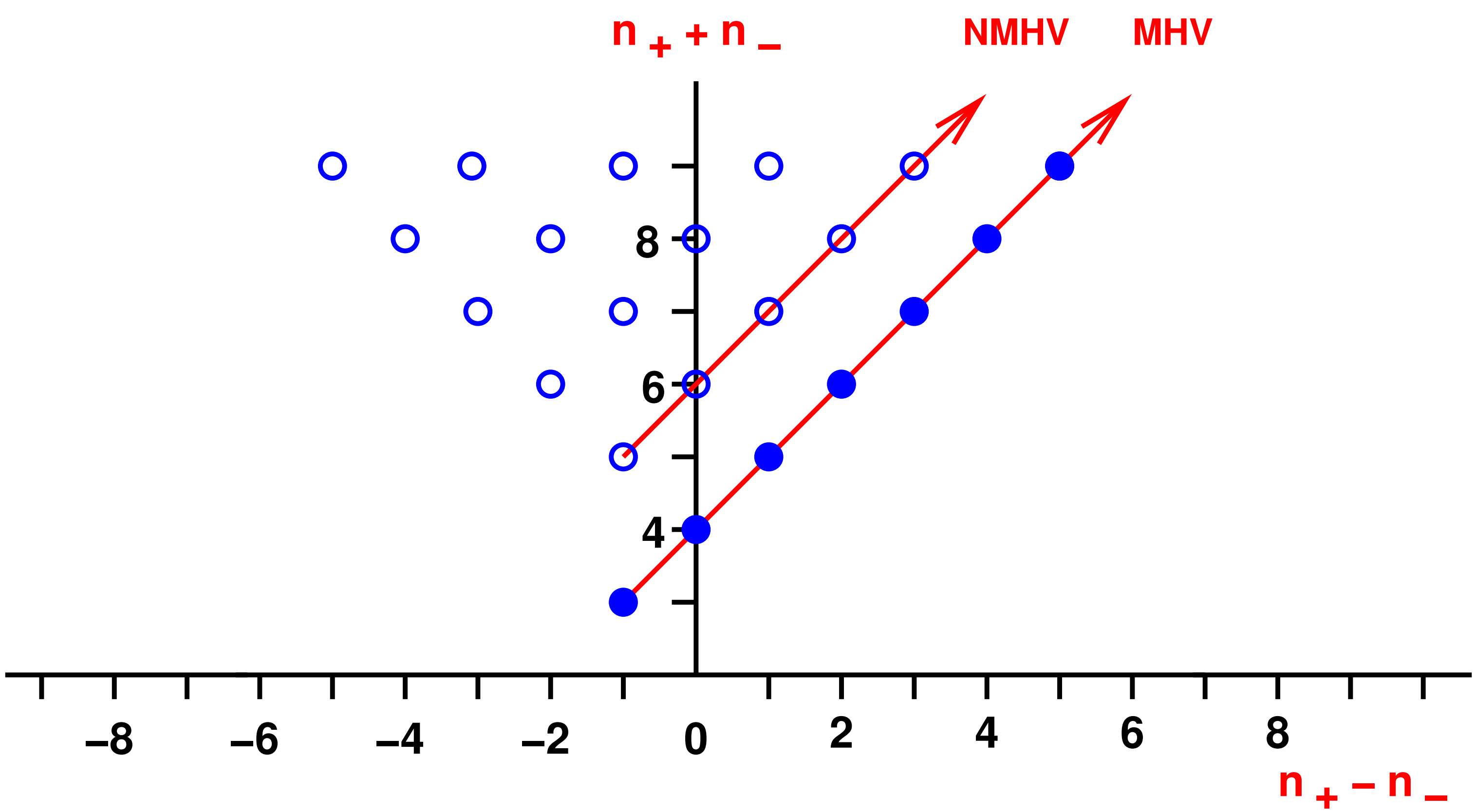
$$+ \sum_{i=4}^{n} \frac{\langle 12\rangle^3}{\langle 2(3,i)\rangle\langle (3,i)i+1\rangle \ldots \langle n1\rangle} \frac{1}{s_{3,i}^2} \frac{\langle (3,i)3\rangle^3}{\langle 34\rangle \cdots \langle i-1 i\rangle\rangle\langle i(3,i)\rangle}.$$

where $(k,i) = k + \cdots + i$ and the off-shell continuation is suppressed

$\Rightarrow$ Lorentz invariant and gauge invariant expressions

Generating all the tree amplitudes

Amplitudes with $i-$ and $j+$ helicities



- MHV rules always adds one negative helicity and any number of positive helicities
  $\Rightarrow$ maps out all allowed tree amplitudes

Other processes

MHV rules have been generalised to many other processes

- ✓ with massless fermions - quarks, gluinos

  Georgiou and Khoze; Wu and Zhu; Georgiou, EWNG and Khoze

- ✓ with massless scalars - squarks

  Georgiou, EWNG and Khoze; Khoze

- ✓ with an external Higgs boson

  Dixon, EWNG, Khoze; Badger, EWNG, Khoze

- ✓ with an external weak boson

  Bern, Forde, Kosower and Mastrolia

Has provided new results for n-particle amplitudes
Also useful for studying infrared properties of amplitudes

Birthwright, EWNG, Khoze and Marquard

Processes with fermions

Similar colour decomposition

$$\mathcal{A}_n(1,\ldots,\Lambda_r,\Lambda_s,\ldots,n) = \sum_{perms} (T^{a_1}\ldots T^{a_n})_{r,s} A_n(\Lambda_r,1,\ldots,n,\Lambda_s)$$

MHV amplitude with 2 fermions and $n-2$ gluons

$$A_n(g_t^-,\Lambda_r^-,\Lambda_s^+) = \frac{\langle tr\rangle^3\langle ts\rangle}{\prod_{i=1}^n \langle i\; i+1\rangle}$$

MHV amplitude with 4 fermions and $n-4$ gluons

$$A_n(\Lambda_r^-,\Lambda_s^+,\Lambda_t^-,\Lambda_u^+) = \frac{\langle rt\rangle^3\langle su\rangle}{\prod_{i=1}^n \langle i\; i+1\rangle}$$

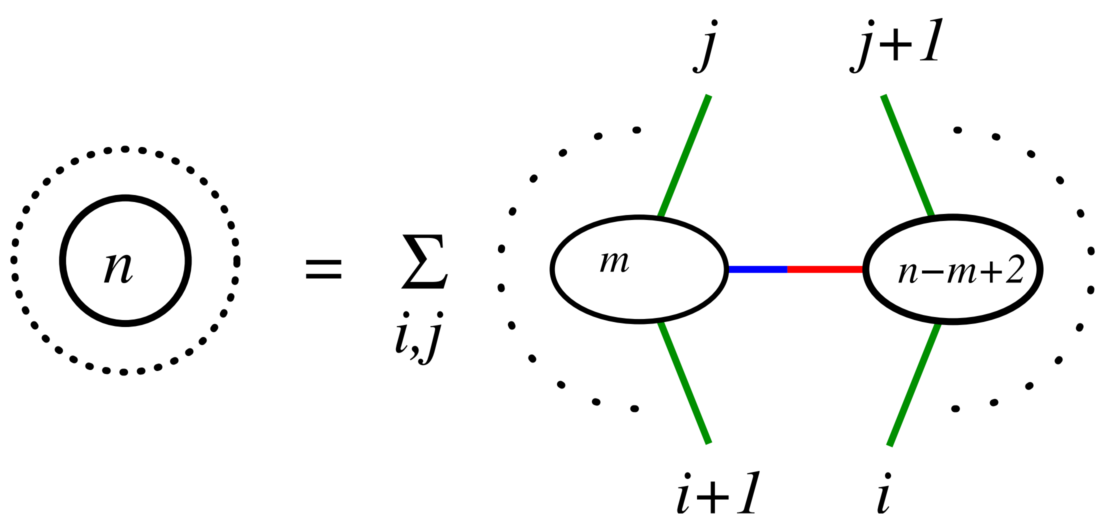
$\Rightarrow$ similar scalar graph construction for fermionic amplitudes

Georgiou and Khoze; Wu and Zhu; Georgiou, EWNG and Khoze

Recursive MHV amplitudes

As the number of negative helicity legs grows, the number of MHV diagrams grows
$\Rightarrow$ Use previously computed on-shell NMHV amplitudes as building blocks for recursion relation

Bena, Bern and Kosower

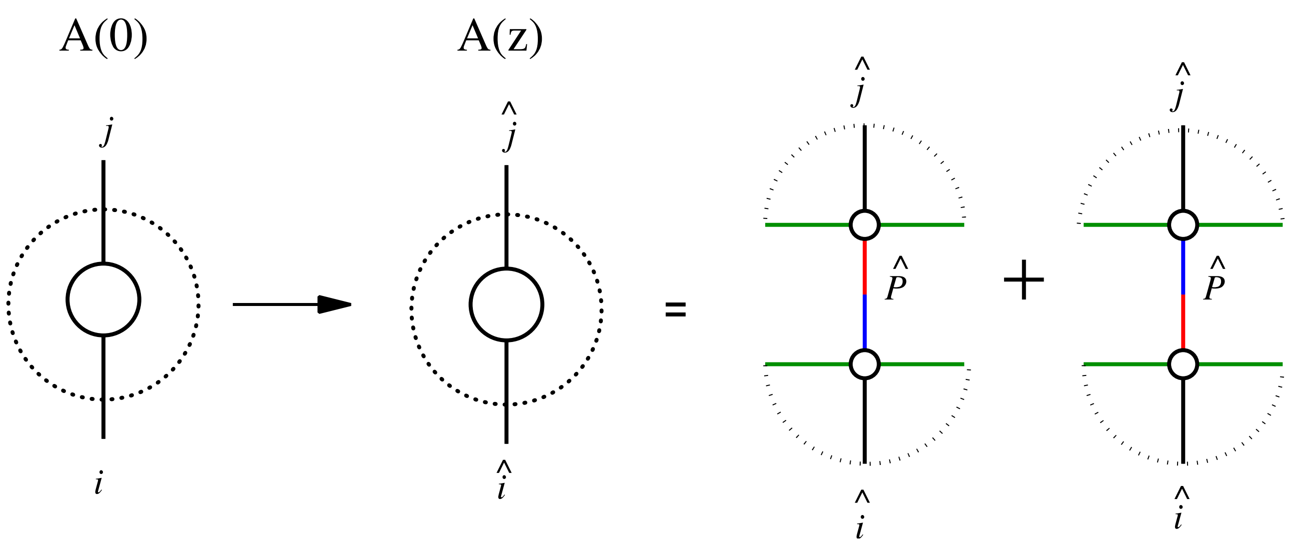


connected by same off-shell continuation as before.
Each blob is an amplitude with fewer particles and fewer negative helicities.
$\Rightarrow$ easily programmed

BCFW on-shell recursion relations

Britto, Cachazo, Feng; Roiban, Spradlin, Volovich

Lets consider an n particle amplitude $A(0)$.



hatted momenta are shifted to put on-shell

$$\hat{i} = i + z\eta, \qquad \hat{j} = j - z\eta, \qquad \hat{P} = P + z\eta$$

$\Rightarrow$ each vertex is an on-shell amplitude

BCFW recursion relations

- It turns out that the shift η is not a momentum, but

$$\eta = \lambda_i \tilde{\lambda}_j \qquad OR \qquad \eta = \lambda_j \tilde{\lambda}_i$$

- The parameter z is fixed by $\hat{P}^2 = 0$

$$z = \frac{P^2}{\langle j|P|i]}$$

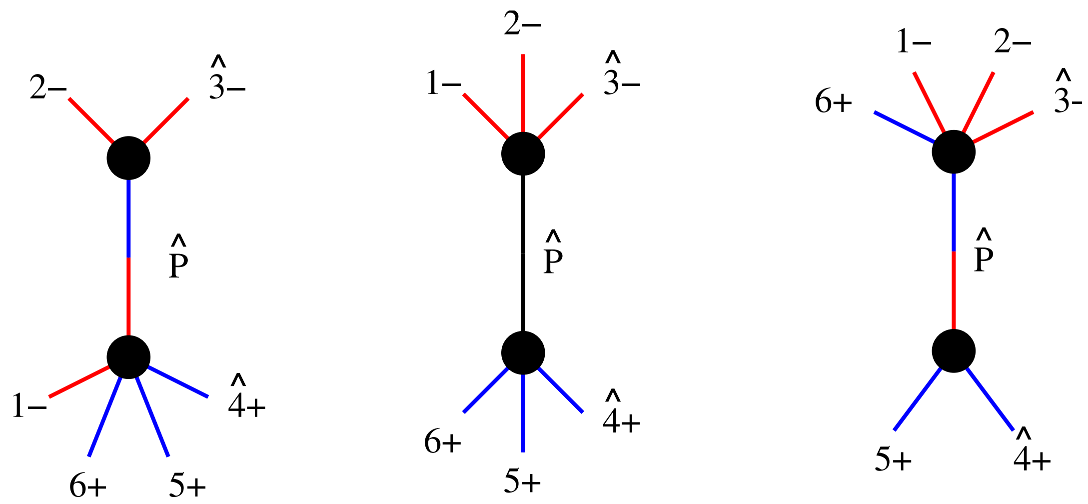
- Easy to prove that by complex analysis based on fact that only simple poles in z occur and that $A(z)$ vanishes as $z \longrightarrow \infty$

Britto, Cachazo, Feng and Witten

- Requires on-shell three-point vertex contributions - both MHV and $\overline{\mathrm{MHV}}$.

BCF - six gluon example

If we select 3 and 4 to be the special gluons, there are only three diagrams (for any helicities)



For this helicity assignment, the middle one is zero!.

$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

$$= \frac{1}{\langle 5|\not{3}+\not{4}|2\rangle}\left(\frac{\langle 1|\not{2}+\not{3}|4\rangle^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{234}} + \frac{\langle 3|\not{4}+\not{5}|6\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle s_{345}}\right)$$

Extremely compact (and correct) results for up to 8 gluons

Other processes

BCF recursion relations have been generalised to other processes

- ✓ with massless fermions - quarks, gluinos

  Luo and Wen

- ✓ gravitons

  Bedford, Brandhuber, Spence and Travaglini; Cachazo and Svrcek

There is nothing (in principle) to stop this approach being applied to particles with mass.

- ✓ massive coloured scalars

  Badger, EWNG, Khoze and Svrcek

- ✓ massive vector bosons and heavy quarks

  Badger, EWNG and Khoze

One loop amplitudes

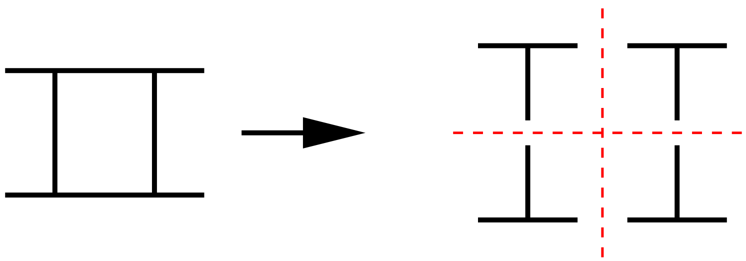
- So far, supersymmetry was not a major factor - tree level amplitudes same for $\mathcal{N}=4$ $\mathcal{N}=1$ and QCD
- Not true at the loop level due to circulating states

$$
\begin{aligned}
A_n^{\mathcal{N}=4} &= A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]} \\
A_n^{\mathcal{N}=1,chiral} &= A_n^{[1/2]} + A_n^{[0]} \\
A_n^{glue} &= A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1,chiral} + A_n^{[0]}
\end{aligned}
$$

- All plus and nearly all-plus amplitudes do not vanish for non-supersymmetric QCD
- A lot of progress by a lot of people

One loop amplitudes

- Key point is that loop amplitudes contain both poles and cuts - e.g. $\log(x)$ has cut for negative x
- Cut contributions are constructible by using unitarity - Cut lines are on-shell and 4-dimensional



Bern, Dixon, Dunbar, Kosower; Britto, Cachazo, Feng

- Pole contributions are amenable to adapted BCFW recursion relations

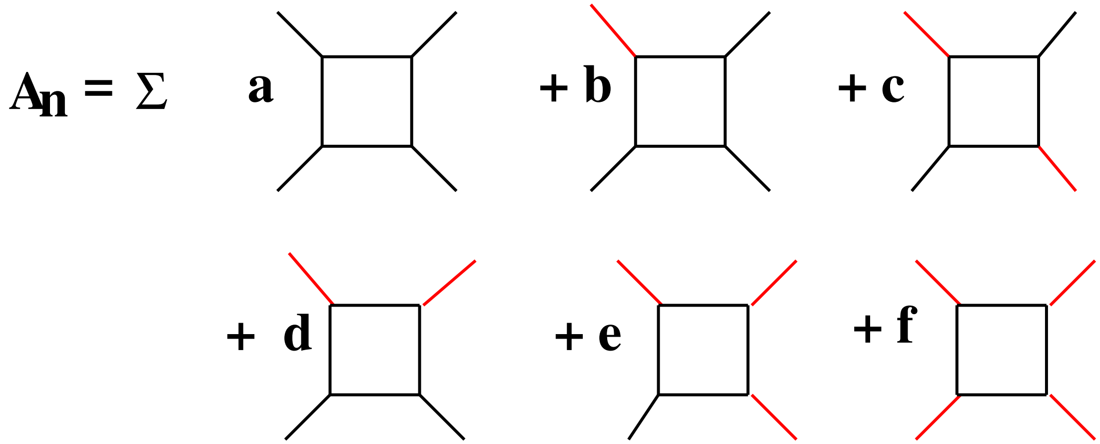
Bern, Dixon, Kosower

SUSY QCD loops

✓ $\mathcal{N}=4$ and $\mathcal{N}=1$ one-loop amplitudes are constructible from their 4-dimensional cuts
$\Rightarrow$ employ unitarity techniques

Bern, Dixon, Dunbar, Kosower

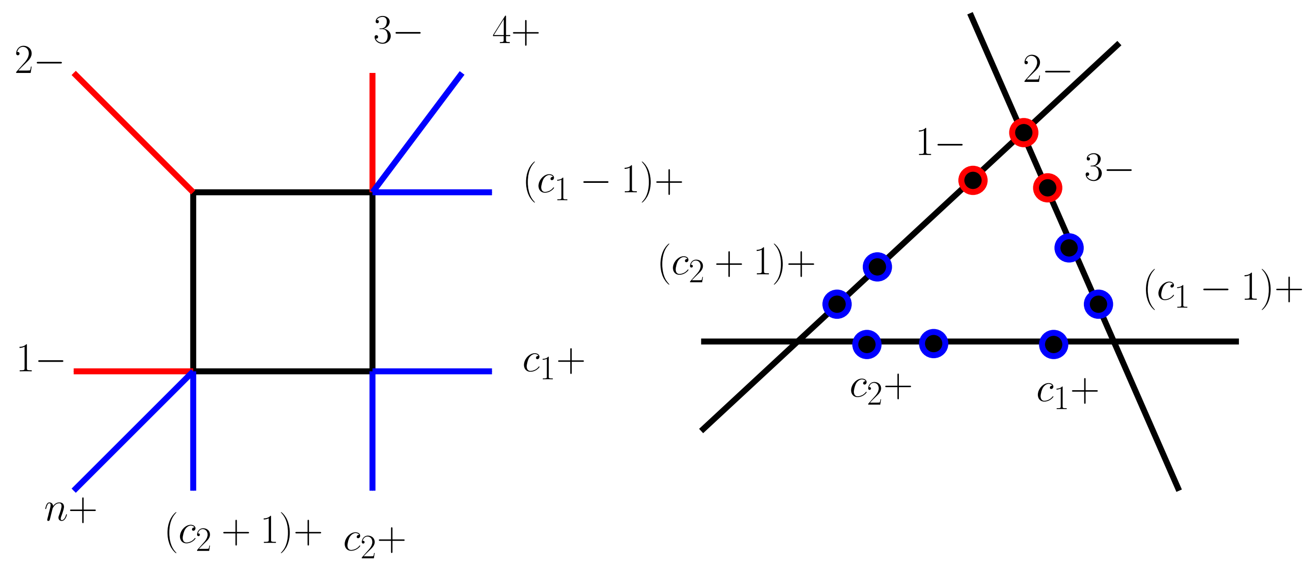
✓ For $\mathcal{N}=4$ all amplitudes are a linear combination of known box integrals



Twistor space interpretation

- Coefficients of boxes have very interesting structures.

  Bern, Del Duca, Dixon, Kosower; Britto, Cachazo, Feng

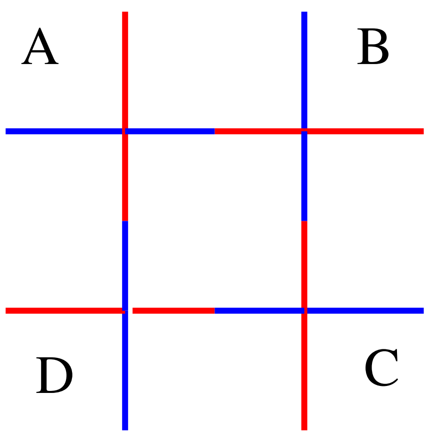


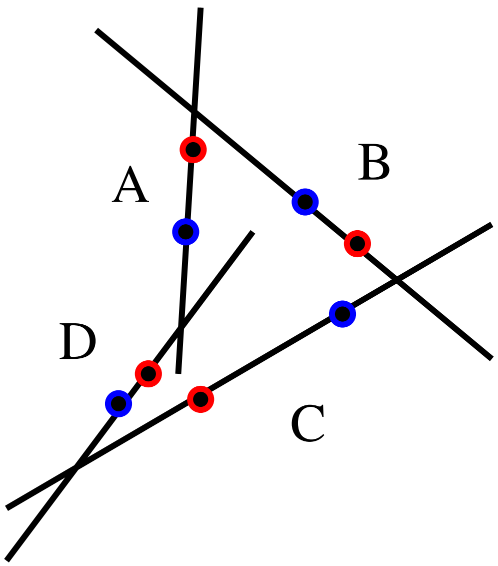
Twistor space interpretation

- Four mass box first appears in eight-point amplitude with four negative and four positive helicities

Bern, Dixon, Kosower

e.g.





QCD loops

QCD amplitudes more complicated

(a) Not 4-dimensional cut constructible. Rational function contribution not probed by 4-d cut

(b) All plus and almost all plus amplitudes not zero - but rational functions. Not protected by SWI.

Nevertheless, all four-point and five-point amplitudes known: Recent progress

✓ On-shell recurrence relations for all plus and almost all plus amplitudes

Bern, Dixon and Kosower, hep-ph/0501240 and 0505055

Recursion relations complicated by double pole terms and boundary terms

✓ Rational parts of infrared divergent amplitudes computed using on-shell recursion relation

Bern, Dixon and Kosower, hep-ph/0507005

Summary

- On-shell techniques are a very exciting and rapidly developing field
- MHV rules for tree-level
  Very simple way of deriving n-point amplitudes for massless partons
- BCFW recursion relations for tree-level
  Very powerful method for deriving amplitudes for both massless and massive particles

  Badger, EWNG, Khoze and Svrcek

- Generalised unitarity and one-loop amplitudes
  SUSY amplitudes cut constructible - coefficients of loop integrals can be *read off* from graphs
  QCD amplitudes contain cut-non constructible parts. These simple pole terms can be attacked using the BCFW relations

  Bern, Dixon, Kosower

  Expect all one-loop six-point gluon amplitudes soon