Reflections on Amplitude Analysis

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Introduction

The two principal reasons for studying meson photoproduction are:

- to determine as accurately as possible the parameters of known baryon resonances, i.e. masses and partial widths, including the radiative widths as they offer a direct probe of hadron structure and can discriminate among models
- to discover new baryon resonances, if any such exist, and to determine their parameters

For both purposes the analyis should be as model-independent as possible. The ideal is to perform a "complete set of experiments", for which pseudoscalar (and scalar) photoproduction are the only feasible reactions.

It is frequently argued that it is necessary to consider vector-meson photoproduction in addition to pseudoscalar photoproduction to resolve the "missing resonance" problem on the grounds that these states do not couple strongly to the πN channel.

However, many of the known resonances couple strongly to the V-N channel (necessarily so, otherwise it would not be possible to access them in photoproduction). As the states are broad and overlapping there must be mixing through these common hadronic channels, so "missing resonances" with the same quantum numbers as the "standard resonances" and not too far apart in mass should be seen in pseudoscalar photoproduction.

It is not possible to perform a "complete set of experiments" for vector-meson photoproduction, although constraints can be placed on the spin observables that should be incorporated directly into data analysis.

C Savkli, F Tabakin & S N Yang, Phys.Rev. C55 (1996) 1132

W M Kloet, W-T Chiang & F Tabakin, Phys.Rev. C58 (1998) 1086

W M Kloet & F Tabakin, Phys.Rev. C61 (2000) 015501

There are non-resonant backgrounds not present in pseudoscalar photoproduction (pomeron exchange, contact term). Data analysis is necessarily model-dependent.

Pseudoscalar photoproduction should give access to most non-strange baryon resonances, although the vector-meson channels cannot necessarily be ignored in the amplitude analysis and certainly cannot be ignored for a full interpretation.

Amplitude Analysis I

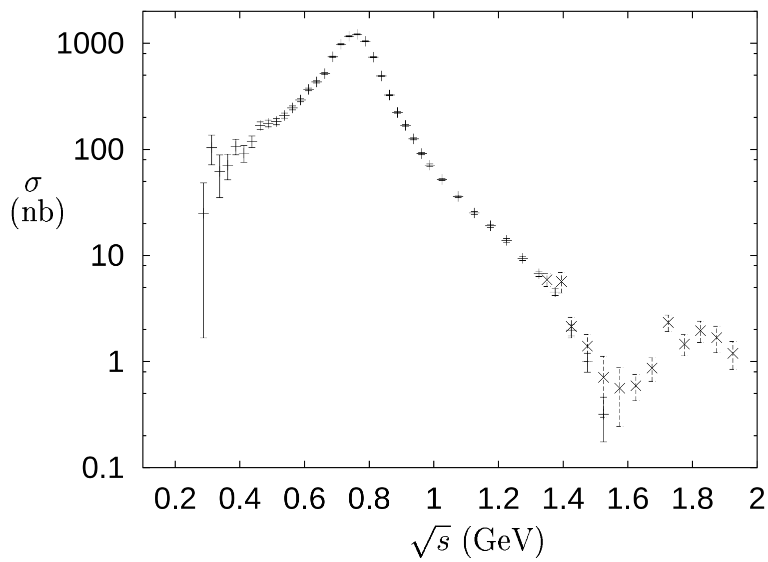
There are basically two approaches to amplitude analysis: energy-independent and energy-dependent.

- The more obvious problems with an energy-independent analysis are ensuring uniqueness at each energy and continuity from energy to energy (there is no *a priori* guarantee that the fit at one energy will match correctly to that at a neighbouring energy) and satisfying the constraints of analyticity (which connects the energy dependence of the real and imaginary parts of a partial-wave amplitude). A less-obvious problem is that of the cut off in the partial-wave expansion, i.e. neglecting all partial waves with $l > l_{\rm max}$. Although the data do not allow the higher partial-waves to be measured, they are non-zero and give a contribution through interference with those partial waves included in the analysis. It is necessary to model these. An obvious example is pion exchange in charged-pion photoproduction. An energy-independent analysis is not quite as model independent as it may seem.
- The problems with an energy-dependent analysis is that it is based on phenomenological parametrizations or a specific dynamical model or some combination of the two. Phenomenological models tend to oversimplify the parametrization of the resonant amplitudes and the non-resonant background that in some cases can be appreciably larger than the resonance contributions. Dynamically-based models build in correlations among partial waves that may not be physical and also tend to oversimplify the parametrization of the resonant amplitudes.

- A multiplicity of models is available for consideration, from dispersion relations, which are the least prescriptive, to a wide variety of dynamical models, some of which are very prescriptive. Isobar models were developed to extract the parameters of the higher-mass nucleon resonances and to incorporate final states other than the pion. K-matrix effective-Lagrangian models were developed initially to study the $\Delta(1232)$ and were later combined with isobar models to perform amplitude analyses as well as extracting resonance parameters. Further progress was made in extracting resonance parameters using the multi-channel K-matrix method combined with the unitary coupled-channel isobar model. Finally dynamical coupled-channel models including off-shell scattering effects, which are claimed to be significant, have been developed.

  A Matsuyama, T Sato and T-S H Lee: nucl-th/0608051

$e^+e^- \to \pi^+\pi^-$: a salutary story



Cross section for $e^+e^- \to \pi^+\pi^-$.

It sounds straightforward to use resonance parametrization in energy-dependent analysis or for extracting resonance parameters, but in practice parametrizing resonances has to be done with great care. Some of the dangers in amplitude analysis are illustrated by the case of the $\rho(1465)$. This state is observed in $e^+e^- \to \pi^+\pi^-$ through interference with the tail of the $\rho(770)$, easily seen in the cross section. The analysis is complicated by the presence of strong thresholds, principally $\omega\pi$ and $a_1\pi$. Passing thresholds changes the total width of a resonance and assumptions have to be made about the magnitude and energy dependence of the partial widths. As a consequence of different assumptions about the high-mass tail of the ρ, the mass of the $\rho(1465)$ varies from 1300 to 1500 MeV and its width from 150 to 500 MeV.

Some history: analyticity

For simplicity consider $\pi\pi$ scattering and construct amplitudes which are symmetric and antisymmetric under crossing ($s \leftrightarrow u$). Take $\pi^+\pi^+$ and $\pi^+\pi^-$ as an example and define

$$\begin{aligned} A_+(\nu,t) &= A(\pi^+\pi^+ \to \pi^+\pi^+) \\ A_-(\nu,t) &= A(\pi^+\pi^- \to \pi^+\pi^-) \end{aligned}$$

where $\nu = \frac{1}{4}(s-u)$. Then the amplitudes which are symmetric and antisymmetric under crossing are

$$\begin{aligned} A^{\rm S}(\nu,t) &= \tfrac{1}{2}\big(A_+(\nu,t) + A_-(\nu,t)\big) \\ A^{\rm A}(\nu,t) &= \tfrac{1}{2}\big(A_+(\nu,t) - A_-(\nu,t)\big) \end{aligned}$$

and they satisfy the fixed-t dispersion relations

$$\begin{aligned} {\rm Re}A^{\rm S}(\nu,t) &= \frac{1}{\pi}P\int_0^\infty d\nu'\frac{{\rm Im}A^{\rm S}(\nu',t)}{(\nu'-\nu)} + (\nu \to -\nu) \\ &= \frac{1}{\pi}P\int_0^\infty d\nu'\frac{2\nu'{\rm Im}A^{\rm S}(\nu',t)}{(\nu'^2-\nu^2)} \\ {\rm Re}A^{\rm A}(\nu,t) &= \frac{1}{\pi}P\int_0^\infty d\nu'\frac{{\rm Im}A^{\rm A}(\nu',t)}{(\nu'-\nu)} - (\nu \to -\nu) \\ &= \frac{2\nu}{\pi}P\int_0^\infty d\nu'\frac{{\rm Im}A^{\rm A}(\nu',t)}{(\nu'^2-\nu^2)} \end{aligned}$$

Strictly the lower limits on the integrals are given by the physical thresholds in the s and u channels but ImA $= 0$ below this limit. Poles can be incorporated by setting ImA $\sim$ δ-function at appropriate values of s or u. Examples of poles are s- and u-channel nucleon exchange in pion-nucleon scattering and s- and u-channel nucleon exchange and t-channel pion exchange in pion photoproduction. Dispersion relations for the partial wave amplitudes may be obtained from those for $A^{\rm S}(\nu,t)$ and $A^{\rm A}(\nu,t)$. The partial-wave dispersion relations which are the ones of direct relevance for partial wave analysis and which have been applied with some success in pion-nucleon scattering and pion photoproduction.

Suppose that the large-ν behaviour of $A^{\rm A}(\nu,t)$ is dominated by a sum of Regge poles for $\nu > \bar{\nu}$:

$$A^{\rm A}(\nu,t) \sim \sum_i \frac{-\pi\beta_i(t)}{\Gamma(\alpha_i(t)+1)\sin(\pi\alpha_i(t))}(1-e^{-i\pi\alpha_i(t)})(2\nu)^{\alpha_i(t)}$$

where the $\alpha_i(t)$ are the Regge trajectories. It is then possible to recast the dispersion relation into a finite-energy sum rule (FESR):

$$\int_0^{\bar{\nu}} d\nu' \, {\rm Im}A^{\rm A}(\nu',t) = -\tfrac{1}{2}\pi \sum_i \beta_i(t)(2\bar{\nu})^{\alpha_i(t)+1}/\Gamma(\alpha_i(t)+2)$$

For an amplitude $A^{\rm S}$ that is even under crossing the corresponding FESR is

$$\int_0^{\bar{\nu}} d\nu' \, \nu'{\rm Im}A^{\rm A}(\nu',t) = -\tfrac{1}{4}\pi \sum_i \beta_i(t)(2\bar{\nu})^{\alpha_i(t)+2}/\Gamma(\alpha_i(t)+3)$$

The FESRs relate the Regge-pole parameters $\beta_i(t)$ and $\alpha_i(t)$ to the low-energy amplitudes. For amplitudes dominated at low energy by resonances this leads to a relation between Regge poles and resonances called duality.

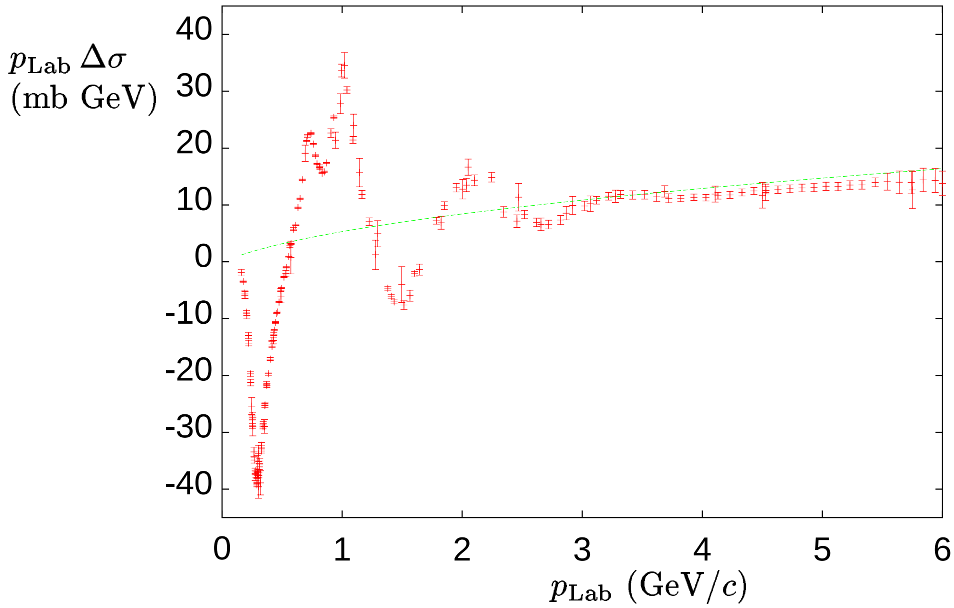
More history: duality

The practical value of FESRs was on putting constraints on the high-energy region, that is using knowledge of the resonance region to help evaluate the Regge terms, notably in pion-nucleon scattering and in pion photoproduction. Unlike the partial-wave dispersion relations the FESRs are not directly relevant for partial-wave analysis (PWA) although they do offer some insight. In particular the roles of resonances and zeros in the amplitude and the connection between the s- and t-channels are illustrated clearly by the concept of duality.

Strictly speaking, the FESRs are only valid for $\bar{\nu}$ sufficiently large for the Regge-pole expression to be a good numerical approximation to the amplitude for $\nu > \bar{\nu}$. However in practice $\bar{\nu}$ has to be taken to be the upper limit of the phase-shift analysis, typically $\sqrt{s} \sim 2$ GeV. If we write the FESR as

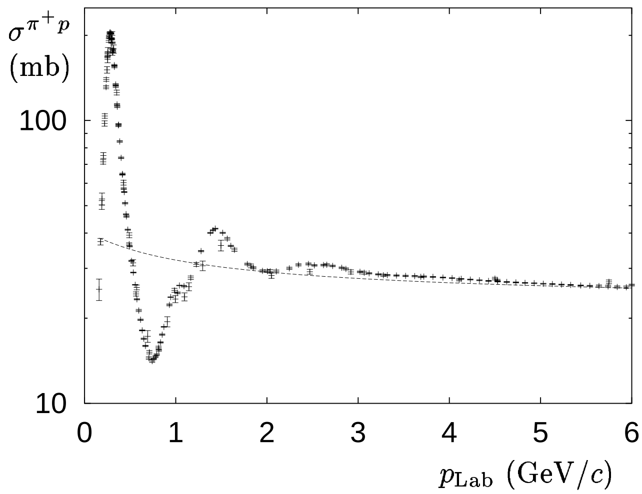
$$\int_0^{\bar{\nu}} d\nu' \, \mathrm{Im}\big(A^{\mathrm{A}}(\nu', t) - A^{\mathrm{A}}_{\mathrm{Regge}}(\nu', t)\big) = 0$$

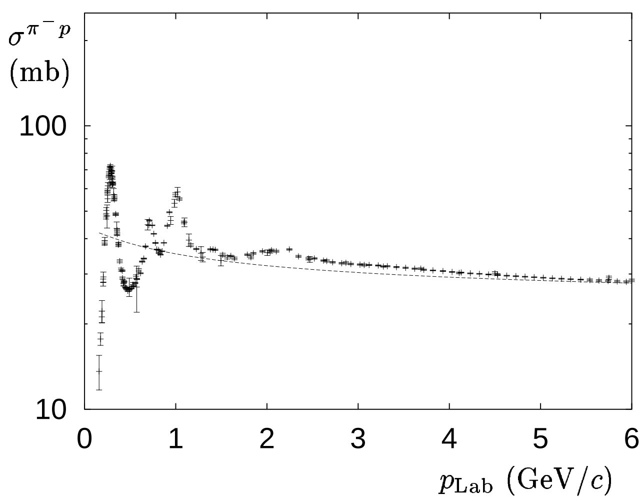
this means that $A^{\mathrm{A}}_{\mathrm{Regge}}(\nu, t)$ describes the amplitude at low energy on the average, provided that the the averaging takes place over intervals smaller than $[0, \bar{\nu}]$. This does happen in practice, as seen in the comparison of the experimental $p_{\mathrm{Lab}}(\sigma^{\mathrm{Tot}}(\pi^- p) - \sigma^{\mathrm{Tot}}(\pi^+ p))$ with an extrapolated Regge fit.



The $\pi^- p$ and $\pi^+ p$ elastic scattering amplitudes receive equal contributions from pomeron exchange, which cancels in the difference, so the non-pomeron t-channel Regge exchanges are dual to the s-channel resonances. This is assumed to be so for $\pi^- p$ and $\pi^+ p$ scattering separately. The extrapolations to low energy of the Regge fits to the high-energy total cross sections gives a good description of the low-energy cross sections on average. For both reactions, the resonances sit on a non-resonant background. This leads us to the assumption that pomeron exchange is dual to the low-energy s-channel non-resonant background. This is two-component duality, and is also true for the total photoproduction cross section.

Note that duality is not a precise concept. Although approximate, the concept does make explicit the fallacy of adding s-channel, u-channel and t-channel contributions as this is certainly double counting. A model is not required to see this as it is a general consequence of analyticity.



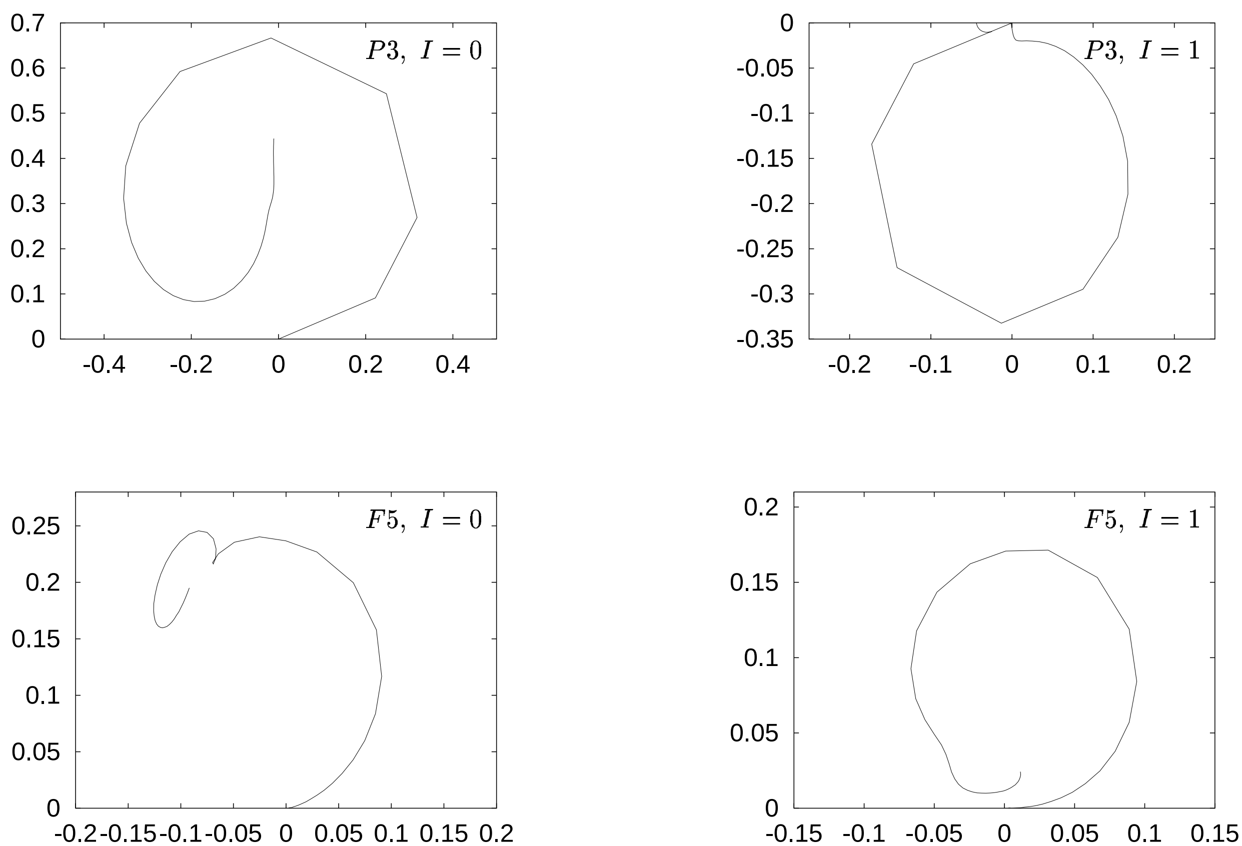


An explicit realisation of two-component duality can be seen in the πN partial-wave amplitudes.

The combinations

$$f^0_{l\pm} = \tfrac{1}{3}\big(f^{\frac{1}{2}}_{l\pm} + 2f^{\frac{3}{2}}_{l\pm}\big) \qquad f^1_{l\pm} = \tfrac{1}{3}\big(f^{\frac{1}{2}}_{l\pm} - f^{\frac{3}{2}}_{l\pm}\big) \tag{1}$$

correspond to isospin 0 and isospin 1 exchange in the t-channel. As the pomeron does not contribute to the t-channel $I = 1$ exchange amplitude, two-component duality predicts that the $f^1_{l\pm}$ should be given entirely by s-channel resonances. On the other hand, the $f_{l\pm}$ should not be given by s-channel resonance alone, but have a predominantly-imaginary smooth background on which the s-channel resonances are superimposed. With the exception of the S-waves, this is what πN partial-wave analysis shows. The P_3 and F_5 partial waves are typical examples.

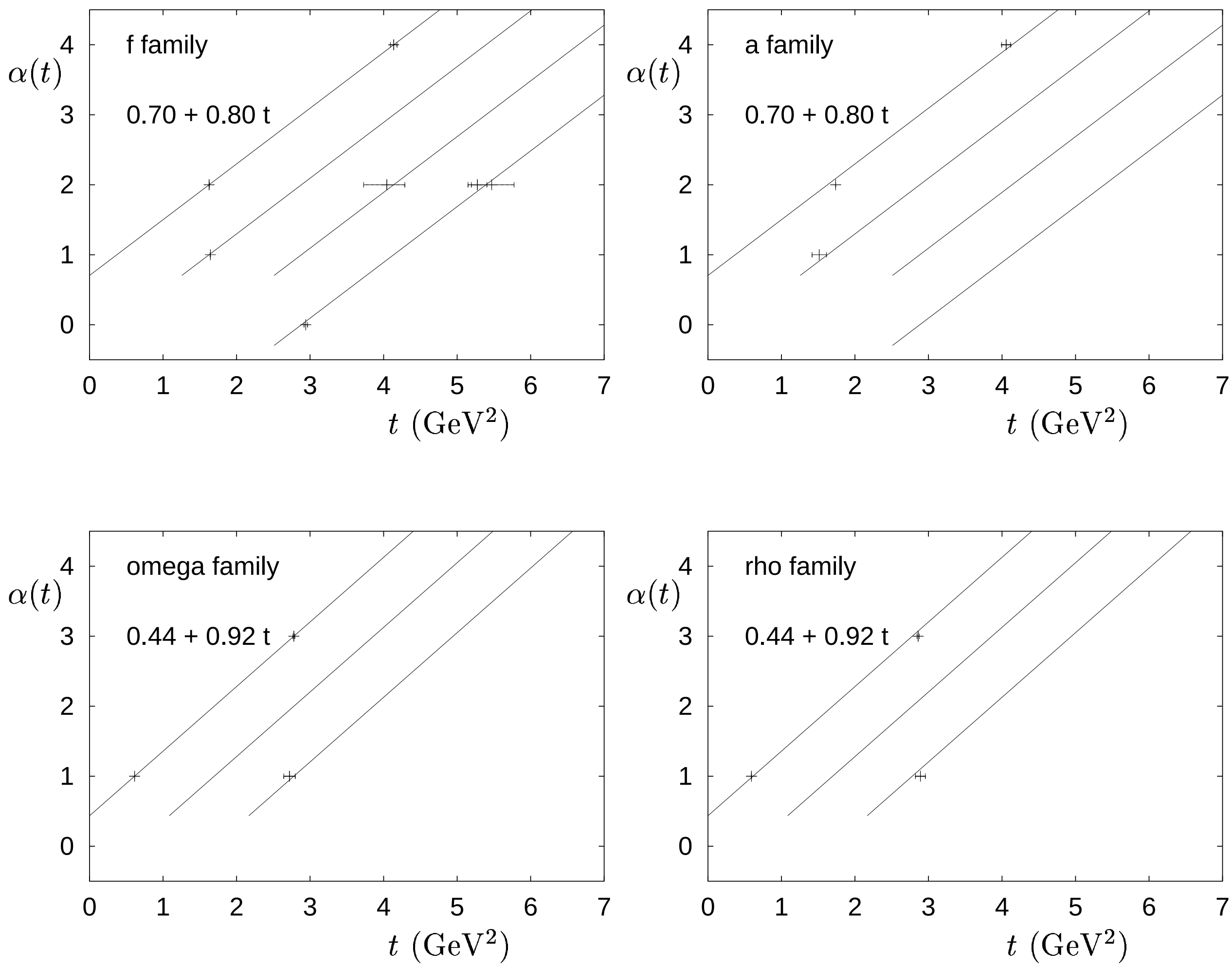


The Veneziano Model

The Veneziano model for the non-pomeron part of the scattering amplitude is crossing symmetric, analytic, has Regge behaviour, satisifies FESRs, contains resonances and exhibits duality. However it does not satisfy unitarity. For a reaction which is identical in all three channels, for example $\pi^0\pi^0$ scattering, the Veneziano amplitude is

$$\begin{aligned} A(s,t) \;=\; & \beta\big(B(-\alpha(s),-\alpha(t)) + B(-\alpha(s),-\alpha(u)) \\ & + B(-\alpha(t),-\alpha(u))\big) \end{aligned}$$

which is explicitly $s \leftrightarrow t$, $s \leftrightarrow u$, $t \leftrightarrow u$ crossing symmetric. $B(x,y)$ is the Euler Beta function $\Gamma(x)\Gamma(y)/\Gamma(x+y)$, β is a constant and $\alpha(s)$ is a real linear trajectory $\alpha(s) = \alpha(0) + \alpha' s$. The model contains an infinite set of daughter trajectories and zeros. The latter arise from the poles in the denominator Gamma functions at, for example, $\alpha(s) + \alpha(t) =$ integer and hence at $s + t =$ constant for linear trajectories. Daughter trajectories are now well established and were predicted by Regge theory long before the lower-lying mesons were discovered.



Four families of daughter Regge trajectories. Only PDG "four-star" resonances have been included.

Barrelet zeros

As $B(x, y)$ may be written as a sum of poles in either variable

$$\begin{aligned} B(x,y) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{x+n} \frac{\Gamma(y)}{\Gamma(y-n)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{y+n} \frac{\Gamma(x)}{\Gamma(x-n)} \end{aligned}$$

we may rewrite the first term in the amplitude as

$$\begin{aligned} B(-\alpha(s), -\alpha(t)) &= \sum_{n=0}^{\infty} \frac{R_n(t)}{n-\alpha(s)} \\ &= \sum_{n=0}^{\infty} \frac{R_n(s)}{n-\alpha(t)} \end{aligned}$$

with R_n a polynomial of degree n:

$$R_n(t) = \frac{1}{n!}(\alpha(t)+1)(\alpha(t)+2)\cdots(\alpha(t)+n)$$

The zeros of the Veneziano model are in fact of greater generality, and depend only on the standard assumptions about analyticity. They are very relevant for partial-wave analysis as they identify multiple ambiguities, first stressed by Barrelet and are known as Barrelet zeros.

The Veneziano model is characterized by the trajectories of its zeros which are straight lines parallel, for example, to $s + t = 0$, passing through the intersections of the two pole sets $s = s_i$, $t = t_i$. In practice the zero trajectories are neither parallel nor linear and they have imaginary parts, but they retain the fundamental properties of the Veneziano model:

(i) they are regular in the Mandelstam (s-t) plane
(ii) they determine the spin of the resonances
(iii) they pass through the intersection of each couple of resonance lines in crossed channels.

The zeros occur in complex-conjugate pairs. The form of the coefficient $R_n(t)$ implies that there is an infinity of zeros, but in practice the relevant number is finite as the order of partial waves which are allowed to contribute is restricted to $l \leq l_{\max}$. Let $z = \cos\theta$. The relation between the differential cross section and the amplitude, $\sigma(z) = |f(z)|^2$, is extended to complex values of z by

$$\sigma(z) = f(z) \times f^*(z^*)$$

The simplest solution, $f(z) = \sigma(z)^{\frac{1}{2}}$, is not appropriate as each zero of σ becomes a branch point of $\sigma^{\frac{1}{2}}$, and $f(z)$ is analytic in the z-plane, except on the right- and left-hand cuts. So let us write

$$\sigma(z) = \sigma_0(z) \prod_{i=1}^{n} (z - z_i)(z - z_i^*) = |\sigma_0(z)^{\frac{1}{2}} \exp(i\phi(z)) \prod_{i=1}^{n} (z - z_i)|^2$$

where $\sigma_0(z)$ is analytic in the z-plane and $\phi(z)$ is an arbitrary phase. The phase is constrained by the optical theorem and elastic unitarity, but there remain 2^n discrete ambiguities corresponding to the number of ways of exchanging some of the z_i with their complex conjugates. Barrelet obtained the zeros for πN scattering and showed that some solutions jumped from one set of zeros to another despite the partial-wave amplitudes appearing to be continuous. So given these discrete ambiguities how was it possible to write a paper with the title "Evidence from πp phase shift analysis for nine more possible nucleon resonances"?

Amplitude analysis II

Provided the energy is not too high then elastic unitarity holds. For elastic scattering this means that the partial-wave amplitudes can be written as

$$f_l(s) = e^{i\delta_l} \sin \delta_l$$

with δ_l real, and for photoproduction the amplitudes can be written as

$$M_l(s) = |M_l| e^{i\delta_l}$$

where δ_l is the corresponding elastic partial-wave phase shift, by Watson's theorem. That is, in both cases the partial-wave amplitude is defined by one parameter, not two. Also in elastic scattering the optical theorem provides an additional constraint at $t = 0$

There are also theoretical aids. For example threshold theorems, as in $\gamma p \to \pi^0 p$ or $\pi\pi$ scattering, which provide a fixed point of departure. Or the results of partial-wave dispersion relations which can provide constraints. It is essential that any PWA reflects correctly the known analytic structure of the amplitudes, for example the nucleon and pion poles in pion photo- and electroproduction. There are also fortunate experimental circumstances, such as the dominance of known resonances. The impact of small partial waves is enhanced by interference with these and the number of intrinsic ambiguities reduced or even removed. Experimental data on different channels can help. A two-fold ambiguity in $\pi\pi$ scattering at $\sqrt{s} \sim 1$ GeV arose because it is sufficiently far from the ρ peak for the ρ to be no longer the dominant amplitude. This was resolved by including $\pi\pi \to K\bar{K}$ in a multichannel analysis.

Elastic unitarity becomes progressively of less relevance as the energy increases, but continues to hold reasonably well for small partial waves.

Of course there is no *a priori* guide to the breakdown of elastic unitarity in any given partial wave: it is a matter of trial and error. For much of the relevant energy region there is still useful input from dispersion relations. They provide information on high partial waves which cannot be determined by the data but which nevertheless have an impact on the analysis through interference and they can be used to ensure that the constraints of analyticity are satisfied. There is also the fortunate experimental fact that the cross sections in $\pi\pi$ and πN scattering and in $\gamma^{(*)}N \rightarrow \pi N$ are dominated by a small number of leading resonances and in elastic scattering the optical theorem again provides a useful constraint at $t = 0$. Further, particular inelastic channels can be dominated by a small number of resonances, for example in $\pi N \rightarrow \eta N$ and $\gamma N \rightarrow \eta N$. As in the case of the two-fold ambiguity in $\pi\pi$ scattering. coupled-channel analysis provides a powerful tool.

Example of πN analysis

It is instructive to consider the procedure used in an early example of πN PWA. All partial waves for $l > l_{\rm max}$ were given by dispersion relations. Energy-independent fits were then performed for all $l \leq l_{\rm max}$ and partial-wave dispersion relations used to smooth the phase shifts and eliminate spurious behaviour, for example violations of causality or analyticity. Constrained energy-independent fits, with big errors on the results of the dispersion relations, were then performed for all $l \leq l_{\rm max}$. The data reject those dispersion-relation predictions they do not like and the fits are still smoothed by the acceptable ones. The procedure is then iterated. This general approach is still being followed today. To use the dispersion relations, every partial wave is represented by a series of complex functions each with the correct analytic and threshold behaviour on the right-hand cut and each multiplied by an adjustable real parameter. To them was added the nucleon poles and the known nearby left-hand cut contribution plus several poles with adjustable residues to represent the distant left-hand cut. The sum was then fitted to the experimental partial-wave amplitudes, the number of right-hand terms required being found from statistical tests. This represented the main problem: too few and the genuine structure may be lost; too many and the fits become unstable. This is similar for any smoothing process, for example polynomial fits to differential cross sections, but differs in that the smoothing is for analyticity as well as continuity.

The comparison of the results of the 1967 analysis with the 2000 PDG compilation shows how successful the initial analyses were. Some states were missed in the 1967 analysis because they were narrow and/or weakly coupled to πN. For example the $D_{13}(1700)$ which is three-star, but $\Gamma = 100$ MeV, $\Gamma_{\pi N}/\Gamma_{Tot} = 0.05$ to 0.15 and the $P_{31}(1750)$ for which $\Gamma = 300$ MeV, but $\Gamma_{\pi N}/\Gamma_{Tot} = 0.05$ to 0.10 and even today is only listed as one-star. The $S_{11}(1535)$ and $S_{11}(1650)$ were combined into a single broad $S_{11}(1591)$. The 1967 analysis did not have the advantage of the high-quality ηN data now available. With hindsight there is evidence in the unsmoothed fits for $P_{13}(1720)$ (four-star) and possibly for the $P_{33}(1920)$, which were removed by the smoothing procedure!

State	Mass	Width	$\Gamma_{\pi N}/\Gamma_{\mathrm{Tot}}$	Rating	Mass	Width	$\Gamma_{\pi N}/\Gamma_{\mathrm{Tot}}$
P_{11}	1440	350	0.6 to 0.7	4*	1466	211	0.66
D_{13}	1520	120	0.5 to 0.6	4*	1541	149	0.51
S_{11}	1535	150	0.35 to 0.55	4*	1591	268	0.70
S_{11}	1650	150	0.55 to 0.90	4*			
D_{15}	1675	150	0.4 to 0.5	4*	1678	173	0.14
F_{15}	1680	130	0.6 to 0.7	4*	1687	177	0.56
D_{13}	1700	100	0.05 to 0.15	3*			
P_{11}	1710	100	0.1 to 0.2	3*	1750	327	0.32
P_{13}	1720	150	0.1 to 0.2	4*			
P_{13}	1900	?	0.2 to 0.3	2*	1863	296	0.21
F_{17}	1990	?	0.05 to 0.10	2*	1983	225	0.13
P_{33}	1600	350	0.10 to 0.25	3*	1688	281	0.10
S_{31}	1620	150	0.2 to 0.3	4*	1635	177	0.28
D_{33}	1700	300	0.1 to 0.2	4*	1691	269	0.14
P_{31}	1750	300	0.05 to 0.10	1*			
S_{31}	1900	200	0.1 to 0.3	2*			
F_{35}	1905	350	0.05 to 0.15	4*	1913	350	0.16
P_{31}	1910	250	0.15 to 0.30	4*	1934	340	0.30
P_{33}	1920	200	0.05 to 0.20	3*			
D_{35}	1930	350	0.1 to 0.2	3*	1954	311	0.15
D_{33}	1940	?	0.1 to 0.3	1*			
F_{37}	1950	300	0.35 to 0.40	4*	1946	221	0.39

Results from 2000 (left-hand) and from 1967 (right-hand)

Complete experiments in pseudoscalar photoproduction

I S Barker, A Donnachie & J K Storrow, Nucl.Phys. B95 (1975) 347

G Keaton & R Workman, Phys.Rev. C53 (1996) 1434

W-T Chiang & F Tabakin, Phys.Rev. C55 (1997) 2054

- Although the result can be stated economically without any formalism, it is useful to define amplitudes. The relevant ones for the derivation are the transversity amplitudes $b_1 \cdots b_4$.

- The notation for experiments is $\{P_\gamma; P_T; P_R\}$ where
  $P_\gamma =$ beam polarisation, $L(\theta)$ linear, C circular
  $P_T =$ direction of target polarisation
  P_R component of recoil polarisation
  with the z-axis in the beam direction, the y-axis normal to the production plane, z' in the direction of the meson and $\boldsymbol{x}' = \boldsymbol{y} \times \boldsymbol{z}'$.

Symbol	Transversity representation	Experiment required	Type
$d\sigma/dt$	$\lvert b_1\rvert^2+\lvert b_2\rvert^2+\lvert b_3\rvert^2+\lvert b_4\rvert^2$	$\{-;-;-\}$	S
$\Sigma d\sigma/dt$	$\lvert b_1\rvert^2+\lvert b_2\rvert^2-\lvert b_3\rvert^2-\lvert b_4\rvert^2$	$\{L(\frac{1}{2}\pi,0);-;-\}$	
$Td\sigma/dt$	$\lvert b_1\rvert^2-\lvert b_2\rvert^2-\lvert b_3\rvert^2+\lvert b_4\rvert^2$	$\{-;y;-\}$	
$Pd\sigma/dt$	$\lvert b_1\rvert^2-\lvert b_2\rvert^2+\lvert b_3\rvert^2-\lvert b_4\rvert^2$	$\{-;-;y\}$	
$Gd\sigma/dt$	$2\ \mathrm{Im}(\mathrm{b}_1\mathrm{b}_3^*+\mathrm{b}_2\mathrm{b}_4^*)$	$\{L(\pm\frac{1}{4}\pi);z;-\}$	BT
$Hd\sigma/dt$	$-2\ \mathrm{Re}(\mathrm{b}_1\mathrm{b}_3^*-\mathrm{b}_2\mathrm{b}_4^*)$	$\{L(\pm\frac{1}{4}\pi);x;-\}$	
$Ed\sigma/dt$	$-2\ \mathrm{Re}(\mathrm{b}_1\mathrm{b}_3^*+\mathrm{b}_2\mathrm{b}_4^*)$	$\{C;z;-\}$	
$Fd\sigma/dt$	$2\ \mathrm{Im}(\mathrm{b}_1\mathrm{b}_3^*-\mathrm{b}_2\mathrm{b}_4^*)$	$\{C;x;-\}$	
$O_xd\sigma/dt$	$-2\ \mathrm{Re}(\mathrm{b}_1\mathrm{b}_4^*-\mathrm{b}_2\mathrm{b}_3^*)$	$\{L(\pm\frac{1}{4}\pi);-;x'\}$	BR
$O_zd\sigma/dt$	$-2\ \mathrm{Im}(\mathrm{b}_1\mathrm{b}_4^*+\mathrm{b}_2\mathrm{b}_3^*)$	$\{L(\pm\frac{1}{4}\pi);-;z'\}$	
$C_xd\sigma/dt$	$2\ \mathrm{Im}(\mathrm{b}_1\mathrm{b}_4^*-\mathrm{b}_2\mathrm{b}_3^*)$	$\{C;-;x'\}$	
$C_zd\sigma/dt$	$-2\ \mathrm{Re}(\mathrm{b}_1\mathrm{b}_4^*+\mathrm{b}_2\mathrm{b}_3^*)$	$\{C;-;z'\}$	
$T_xd\sigma/dt$	$2\ \mathrm{Re}(\mathrm{b}_1\mathrm{b}_2^*-\mathrm{b}_3\mathrm{b}_4^*)$	$\{-;x;x'\}$	TR
$T_zd\sigma/dt$	$2\ \mathrm{Im}(\mathrm{b}_1\mathrm{b}_2^*-\mathrm{b}_3\mathrm{b}_4^*)$	$\{-;x;z'\}$	
$L_xd\sigma/dt$	$2\ \mathrm{Im}(\mathrm{b}_1\mathrm{b}_2^*+\mathrm{b}_3\mathrm{b}_4^*)$	$\{-;z;x'\}$	
$L_xd\sigma/dt$	$2\ \mathrm{Re}(\mathrm{b}_1\mathrm{b}_2^*+\mathrm{b}_3\mathrm{b}_4^*)$	$\{-;z;z'\}$	

Resolving ambiguities

- Without considering discrete ambiguities, seven measurements are required to determine the four helicity amplitudes (four magnitudes and three phases) up to an arbitrary overall phase. However it is necessary to resolve all discrete ambiguities to extract complete information.
- In BDS the following rule was promulgated:
  In order to determine amplitudes without discrete ambiguities, one has to measure five double-spin observables along with the four S-type measurements, provided no four double-spin observables are selected from the same set of BT, BR and TR.
- However KW argued that there are cases obeying the BDS rule that still leave an unresolved discrete ambiguity although they were unable to provide sufficient conditions for resolving this
- CT were able to show that **four appropriately chosen** double-spin observables, along with the four S-type measurements, are sufficient to resolve all discrete ambiguities.
- Note that these discrete ambiguities are at the level of the transversity amplitudes. Even once they are removed there remains the problem of discrete ambiguities at the level of partial-wave amplitudes

Complete sets of eight measurements

- Define $A,\ B,\ C,\ D,\ E,\ F$ as sets of pairs of double-spin observables:

$$
\begin{aligned}
\{(H,E),(O_x,C_z),(T_x,L_z)\} &= A\\
\{(G,F),(O_x,C_z),(T_z,L_x)\} &= B\\
\{(H,F),(O_x,C_x),(T_x,T_z)\} &= C\\
\{(G,H),(O_x,O_z),(T_x,L_x)\} &= D\\
\{(E,F),(C_x,C_z),(T_z,L_z)\} &= E\\
\{(G,E),(O_z,C_z),(L_x,L_z)\} &= F
\end{aligned}
$$

and $X,\ Y$ as sets of double-spin observables:

$$
\begin{aligned}
\{H,E,O_x,C_z,T_x,L_z\} &= X\\
\{G,F,O_z,C_x,T_Z,L_x\} &= Y
\end{aligned}
$$

- The CT rules for a complete set are considerably more complicated than the BDS rule. They cannot be expressed succinctly nor is there any simple physical guidance.

$2+2$ cases

Pick one pair of double-spin observables from the same type ($BT,\ BR,\ TR$) and a second pair from another type.

- $2\ BT + 2\ TR$: choose at least one pair from set D or set E
- $2\ BT + 2\ BR$: choose at least one pair from set C or set F
- $2\ BR + 2\ TR$: choose the BR pair from set D or set E and the TR pair from set C or set F

$2 + 1 + 1$ **cases**

Pick one pair of double-spin observables from the same type (BT, BR, TR) and one observable from each of the remaining two types. Most combinations allow ambiguities to be resolved. Those that do **not** are:

- when the pair belongs to set A and the other two belong to set X or to set Y
- when the pair belongs to set B, one of the other two observables belongs to set X (Y) and the fourth to set Y (X).

$3 + 1$ **cases**

Pick three double-spin observables from one type and one observable from another type. Ambiguities cannot be resolved in these cases.

4 **cases**

Pick all double-spin observables from the same type. The amplitudes can never be determined uniquely in these cases.

Bounds

Not all allowed experiments are necessarily practical, for example the asymmetry to be measured may be small. Bounds can be very useful in deciding which experiments to do.

- All double-spin observables are bounded by the type S:

$$|X_{BT}| \leq \min\Big\{\sqrt{1-\Sigma^2}, \sqrt{1-T^2}\Big\}$$

where

$$X_{BT} = G, H, E \text{ or } F$$

$$|X_{BR}| \leq \min\Big\{\sqrt{1-\Sigma^2}, \sqrt{1-P^2}\Big\}$$

where

$$X_{BR} = O_x, O_z, C_x \text{ or } C_z$$

$$|X_{TR}| \leq \min\Big\{\sqrt{1-P^2}, \sqrt{1-T^2}\Big\}$$

where

$$X_{TR} = T_x, T_z, L_x \text{ or } L_z$$

- If one double-spin observable has been measured then there exist more stringent bounds between two observables of a given type and the type S.

$$\max\Big\{(G^2+E^2),(H^2+F^2),(G^2+H^2),(E^2+F^2)\Big\}$$
$$\le \min\Big\{\sqrt{1-\Sigma^2},\sqrt{1-T^2}\Big\},$$
$$\max\Big\{(O_x^2+O_z^2),(C_x^2+C_z^2),(O_x^2+C_x^2),(O_z^2+C_z^2)\Big\}$$
$$\le \min\Big\{\sqrt{1-\Sigma^2},\sqrt{1-P^2}\Big\},$$
$$\max\Big\{(T_x^2+T_z^2),(L_x^2+L_z^2),(T_x^2+L_x^2),(T_z^2+L_z^2)\Big\}$$
$$\le \min\Big\{\sqrt{1-P^2},\sqrt{1-T^2}\Big\}.$$

$$\max\Big\{|G\pm F|,|E\pm H|\Big\}\le 1\pm P,$$
$$\max\Big\{|T_z\pm L_x|,|T_x\pm L_z|\Big\}\le 1\pm \Sigma,$$
$$\max\Big\{|O_x\pm C_z|,|O_z\mp C_x|\Big\}\le 1\pm T.$$

Many other relationships can be derived, for example if three double-spin observables of a particular type are known then the fourth member of that type is uniquely determined and its measurement is redundant. (This is another statement of the inability to determine uniquely all the amplitudes if all four double-spin measurements are of the same type.) Explicit procedures for determining all relationships and bounds are given in CT

Summary

- The CT rules require that at least two of the four double-spin measurements involve measuring the recoil baryon polarization.
- The bounds on the double-spin measurements are absolute and must be taken into account when planning which measurements to make. For example, suppose the polarized beam asymmetry Σ is close to 1 in some kinematical region (which does happen), then

  $$|X_{BT}| \leq \min\left\{\sqrt{1-\Sigma^2}, \sqrt{1-T^2}\right\}$$

  where

  $$X_{BT} = G, H, E \text{ or } F$$

  means that measuring any of the beam-target asymmetries in that kinematic region will provide no new information.
- Note, however, that obtaining a unique set of transversity amplitudes, or equivalently helicity amplitudes, does **not** guarantee a unique set of multipole amplitudes. In principle discrete ambiguities (i.e. more than one solution) remain at the level of the multipole amplitudes (recall Barrelet). Even if a "solution" appears to be continuous in energy it may be that at some point it has switched from one solution to another (i.e. two solutions have crossed). Barrelet demonstrated this for some fits to pion-nucleon scattering. Additional information (or assumptions) may be required.