Effective field theories in a finite volume

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V. Bernard, M. Lage, U.-G. Meißner and AR, arXiv:1010.6018

D. Hoja, U.-G. Meißner and AR, arXiv:1001.1641

Strong interactions: from methods to structures, 14 February 2011, Bad-Honnef

Plan

- Introduction: effective field theories in a finite volume
- Lüscher equation in a moving frame
- The nature of scalar mesons:
  - Identification of the hadronic molecules on the lattice
  - Twisted boundary conditions and the generalization of Lüscher formula for multi-channel scattering
  - Strangeness content of the exotic states
- Infinite-volume limit of the matrix elements on the lattice
- Conclusions, outlook

Goals:

- Extracting scattering S-matrix elements from lattice simulations:
  - Elastic scattering (Lüscher)
  - Inelastic scattering: phase shifts, inelasticities, scattering lengths
  - Resonance parameters
  - The nature of the resonances: $q\bar{q}$ states, hadronic molecules, tetraquarks, etc
- Extracting matrix elements of external currents:
  - Magnetic moments, axial charges, etc
  - Electromagnetic formfactors

Infinite-volume limit?

$\hookrightarrow$ Use effective field theory methods!

Lüscher approach

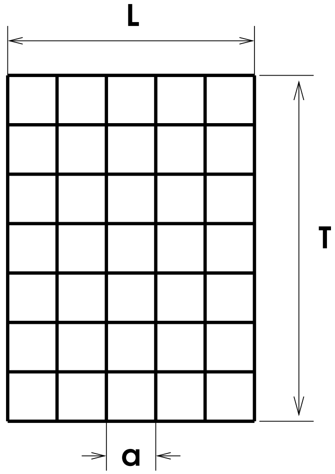
M. Lüscher, lectures given at Les Houches (1988); NPB 364 (1991) 237

. . .

Lattice calculations are always done in a finite volume

$\Rightarrow$ *The spectrum is discrete*

$\Rightarrow$ *scattering? resonances?*



Energy spectrum at: $R_{\rm int}^{-1} L \simeq ML \gg 1\,, \quad a \to 0$

- Momenta are small: $p \simeq 2\pi/L \ll M \quad \Rightarrow \quad$ NR EFT
- Finite-volume corrections to the energy levels are only power-suppressed in L
- Studying the dependence of the energy levels on L gives the scattering phase in the infinite volume $\quad \Rightarrow$ Resonances

Example: Lüscher equation the moving frame

Covariant non-relativistic EFT in the infinite volume, no spin:

G. Colangelo, J. Gasser, B. Kubis and AR, PLB 638 (2006) 187

$$\mathcal{L} = \Psi^\dagger 2W_N(i\partial_t - W_N)\Psi + \Phi^\dagger 2W_\pi(i\partial_t - W_\pi)\Phi + C_0\, \Psi^\dagger\Phi^\dagger\Phi\Psi + \cdots$$

$$W_N = \sqrt{M_N^2 - \triangle}\,, \qquad W_\pi = \sqrt{M_\pi^2 - \triangle}$$

Lippmann-Schwinger equation in an arbitrary moving frame:

$$T(p_1p_2;q_1q_2) = V(p_1p_2;q_1q_2) + \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{V(p_1p_2;k_1k_2)T(k_1k_2;q_1q_2)}{4w_Nw_\pi(w_N + w_\pi - q_1^0 - q_2^0)}$$

$$\mathbf{k}_N = \eta\mathbf{P} + \mathbf{k}\,, \quad \mathbf{k}_\pi = (1-\eta)\mathbf{P} + \mathbf{k}\,, \qquad \eta = \frac{1}{2}\left(1 + \frac{m_N^2 - M_\pi^2}{s}\right)$$

Choice of the interaction Lagrangian:
V depends on the Lorentz-invariant variables $p_i \cdot p_j$

Partial-wave expansion: infinite volume

LS equation for the on-shell partial-wave amplitudes:

$$T_L(s) = V_L(s) + V_L(s)\, G(s)\, V_L(s)$$

$$V_L(s) = (s - s_t)^L \left(\sum_n C_{nL} (s - s_t)^n \right), \qquad s_t = (m_N + M_\pi)^2$$

$$G(s) = \lim_{d \to 3} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{4 w_N w_\pi (w_N + w_\pi - q_1^0 - q_2^0)} = \frac{ip(s)}{8\pi\sqrt{s}}$$

Matching:

C_{nL} $\Leftrightarrow$ effective-range expansion parameters (scattering length, effective radius, etc)

Non-relativistic EFT in a finite volume

S.R. Beane et al., NPA 747 (2005) 55 (no spin, CM frame)

Finite volume: $$\int \frac{d^3\mathbf{k}}{(2\pi)^3}(\cdots) \to \frac{1}{L^3}\sum_{\mathbf{k}}(\cdots), \quad \mathbf{k} = \frac{2\pi\mathbf{n}}{L}, \ \mathbf{n} \in Z^3$$

$\Rightarrow$ Poles of the resolvent $\langle \mathbf{p}|\frac{1}{z-H}|\mathbf{q}\rangle$ $\Rightarrow$ Spectrum in a box

Lüscher formula for the scattering phase shift, moving frame

M. Lage et al., in progress

$$\det\left(\delta_{mm'}\delta_{ll'} - \tan\delta_l(p) \underbrace{\mathcal{M}^{\mathbf{d}}_{lm,l'm'}(p)}_{\text{Free Green function, finite volume}}\right) = 0$$

$$\mathbf{d} = \frac{L}{2\pi}(\mathbf{p}_1 + \mathbf{p}_2)$$

Lüscher approach in the moving frame

The matrix $\mathcal{M}^{\mathbf{d}}_{lm,l'm'}(p)$ is given by a linear combination of

$$Z^{\mathbf{d}}_{lm}(1,q^2) = \lim_{s\to 1} \sum_{\mathbf{r}\in P_d} \frac{Y_{lm}(\mathbf{r})}{(\mathbf{r}^2 - q^2)^s}$$

$$P_d = \left\{\mathbf{r}\in\mathbb{R}^3 | \mathbf{r} = \boldsymbol{\gamma}^{-1}(\mathbf{n} - \eta\mathbf{d}) \quad \text{for some} \quad \mathbf{n}\in\mathbb{Z}^3\right\}$$

$$\boldsymbol{\gamma}^{-1}\mathbf{a} \doteq (\gamma^{-1}a_{\|}, \mathbf{a}_{\perp})\,, \qquad q = \frac{pL}{2\pi}$$

- By using *covariant* NR effective field theory method, we obtain Lüscher formula in the moving frame, including unequal mass case and scattering of the particles with spin
- The rotation symmetry is broken down to (a subgroup of) (the double cover of) the cubic group; mixing of the partial waves
- Symmetry is reduced in the unequal mass case, e.g., $Z^{\mathbf{d}}_{lm}(1,q^2) \neq 0$ for odd values of l
- Planned: using *covariant* NR EFT to describe 3-body systems

Where are the resonance poles?

Consider CM frame:

Suppose that there exists an isolated narrow resonance in the vicinity of the elastic threshold. Assume that effective range expansion for the quantity $p\cot\delta(p)$ is convergent in the resonance region.

$$p\cot\,\delta(p) = A_0 + A_1 p^2 + \cdots$$

- $\Rightarrow$ $A_0, A_1, \cdots$ are measured on the lattice
- $\Rightarrow$ Resonance pole in the complex momentum plane:

$$\cot\,\delta(p_*) = -i$$

- $\Rightarrow$ $p \to p_*$ effectively amounts to the infinite-volume limit $\text{Im}\,q \to \mp\infty$, since

$$\lim_{\text{Im}\,q\to\mp\infty} \frac{Z_{00}(1;q^2)}{\pi^{3/2}q} = \mp i\,\text{sign}(\text{Re}\,q)$$

Example: The nature of the scalar mesons

V. Bernard, M. Lage, U.-G. Meißner and AR, arXiv:1010.6018

- What is the nature of the scalar mesons with a mass ~ 1 GeV:

  $q\bar{q}$ states? tetraquarks? hadronic molecules? $\ldots$?
- What kind of the information can be learned from the lattice?

Consider $f_0(980)$ in the vicinity of "heavy" threshold $s = s_t = 4M_K^2$:

2-channel LS equation, infinite volume ("heavy"=1, "light"=2):

$$\begin{aligned} T_{11} &= H_{11} + H_{11} i q_1 T_{11} + H_{12} i q_2 T_{21} \\ T_{21} &= H_{21} + H_{21} i q_1 T_{11} + H_{22} i q_2 T_{21} \end{aligned}$$

Resonance pole(s) are determined from the secular equation:

$$1 - i q_1 H_{11} - i q_2 H_{22} - q_1 q_2 (H_{11} H_{22} - H_{12}^2) = 0$$

Quark compounds or molecules?

Compositeness criterium: S. Weinberg, PRD **130** (1963) 776; Phys. Rev. **131** (1963) 440; Phys. Rev. **137** (1965) B672

Pole counting: D. Morgan, NPA **543** (1992) 632
N. A. Törnqvist, PRD **51** (1995) 5312
D. Morgan and M. R. Pennington, PLB **258** (1991) 444; PRD **48** (1993) 1185
V. Baru *et al*, PLB **586** (2004) 53
...

S-matrix poles:

- A single pole near threshold: hadronic molecule
- More poles correspond to tightly bound state ($q\bar{q}$? tetraquarks?)
- The argument is only valid near threshold

$\Rightarrow$ Measure $H_{ij}(s)$ on the lattice (multi-channel Lüscher equation)

$\Rightarrow$ Find the position of the pole(s)

Lippmann-Schwinger equation in a finite volume

$$iq_i \to \mathcal{J}_i = \frac{2}{\sqrt{\pi}L} Z_{00}(1;k_i^2)\,, \quad k_i = \frac{Lq_i}{2\pi}\,, \quad i=1,2$$

The pseudophase $\delta(q_2)$ is a measurable on the lattice:

$$\delta(q_2) = -\phi(k_2) + \pi n\,, \qquad k_2 = \frac{q_2 L}{2\pi}\,, \quad \tan\phi(k_2) = -\frac{\pi^{3/2}k_2}{Z_{00}(1;k_2^2)}$$

Expressing pseudophase through H_{ij}:

$$\tan\delta(q_2) = q_2\left(H_{22}(s) + \frac{\mathcal{J}_1(s)(H_{12}(s))^2}{1-\mathcal{J}_1(s)H_{11}(s)}\right)$$

$\Rightarrow \quad \delta(s) \to H_{ij}(s)$: not enough equations in the multi-channel case!

Twisted boundary conditions: $K\bar{K}$ threshold

P.F. Bedaque, PLB 593 (2004) 82;
G.M. de Diviitis, R. Petronzio and N. Tantalo, PLB 595 (2004) 408;
G.M. de Diviitis and N. Tantalo, hep-lat/0409154;
C.T. Sachrajda and G. Villadoro, PLB 609 (2005) 73

The twisting angle: $\theta \in (0, 2\pi)$

$$u(\mathbf{x} + L\mathbf{e}_i) = u(\mathbf{x})\,, \quad d(\mathbf{x} + L\mathbf{e}_i) = d(\mathbf{x})\,, \quad s(\mathbf{x} + L\mathbf{e}_i) = \mathrm{e}^{i\theta} s(\mathbf{x})$$

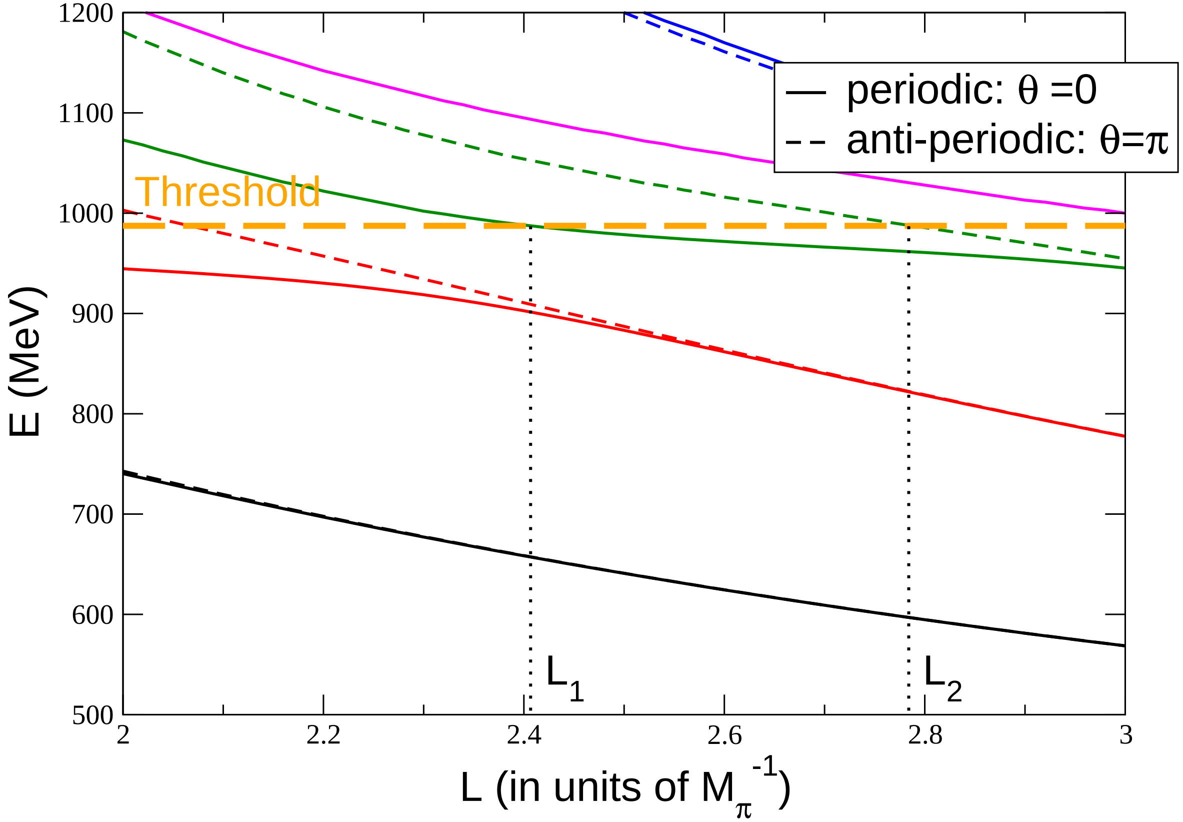
Only the "heavy" threshold with strange particles moves:

$$Z_{00}(1; k_1^2) \to Z_{00}^{\theta}(1, k_1^2) = \frac{1}{\sqrt{4\pi}} \lim_{s\to 1} \sum_{\mathbf{n}\in Z^3} \frac{1}{((\mathbf{n} - \boldsymbol{\theta}/2\pi)^2 - k_1^2)^s}$$

Pseudophase: $$\tan\delta^{\theta}(q_2) = q_2 \left(H_{22}(s) + \frac{\mathcal{J}_1^{\theta}(s)(H_{12}(s))^2}{1 - \mathcal{J}_1^{\theta}(s) H_{11}(s)} \right)$$

$\Rightarrow$ Independent matching conditions for every single θ

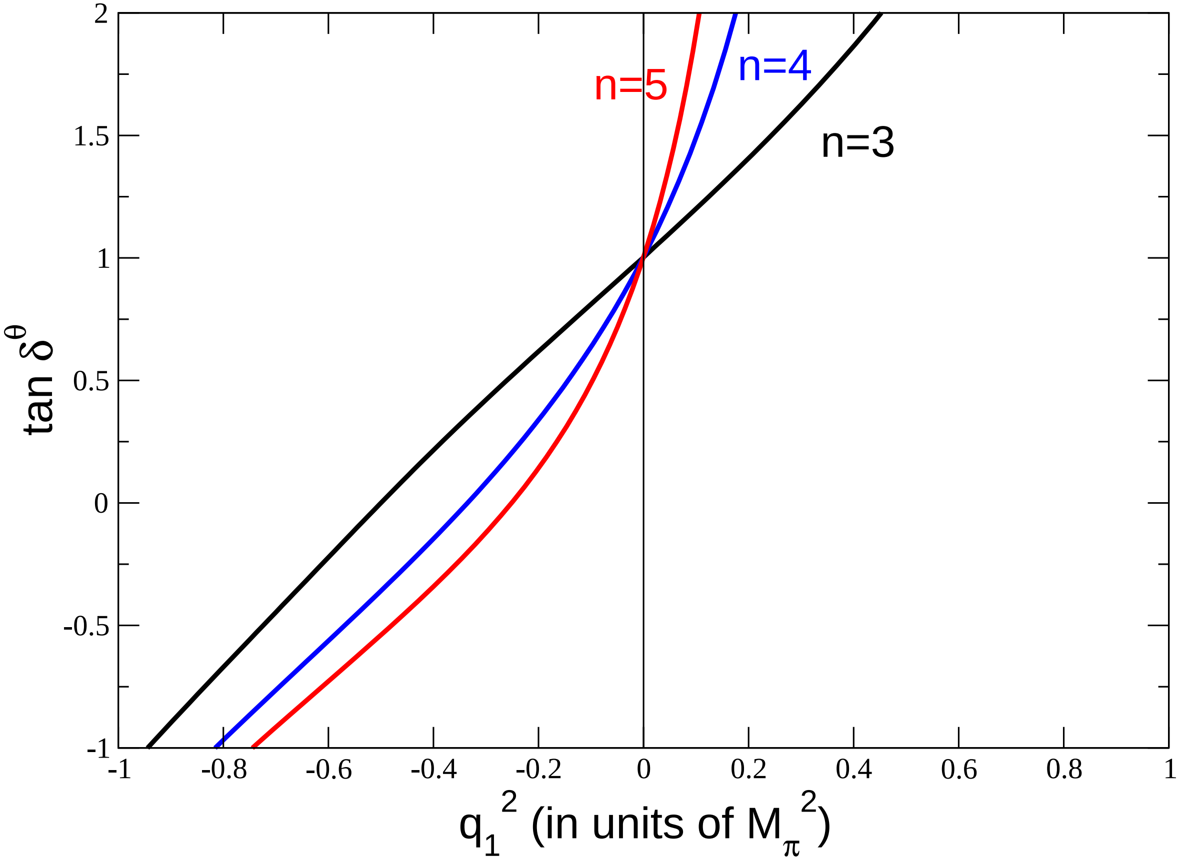
An example of the energy levels



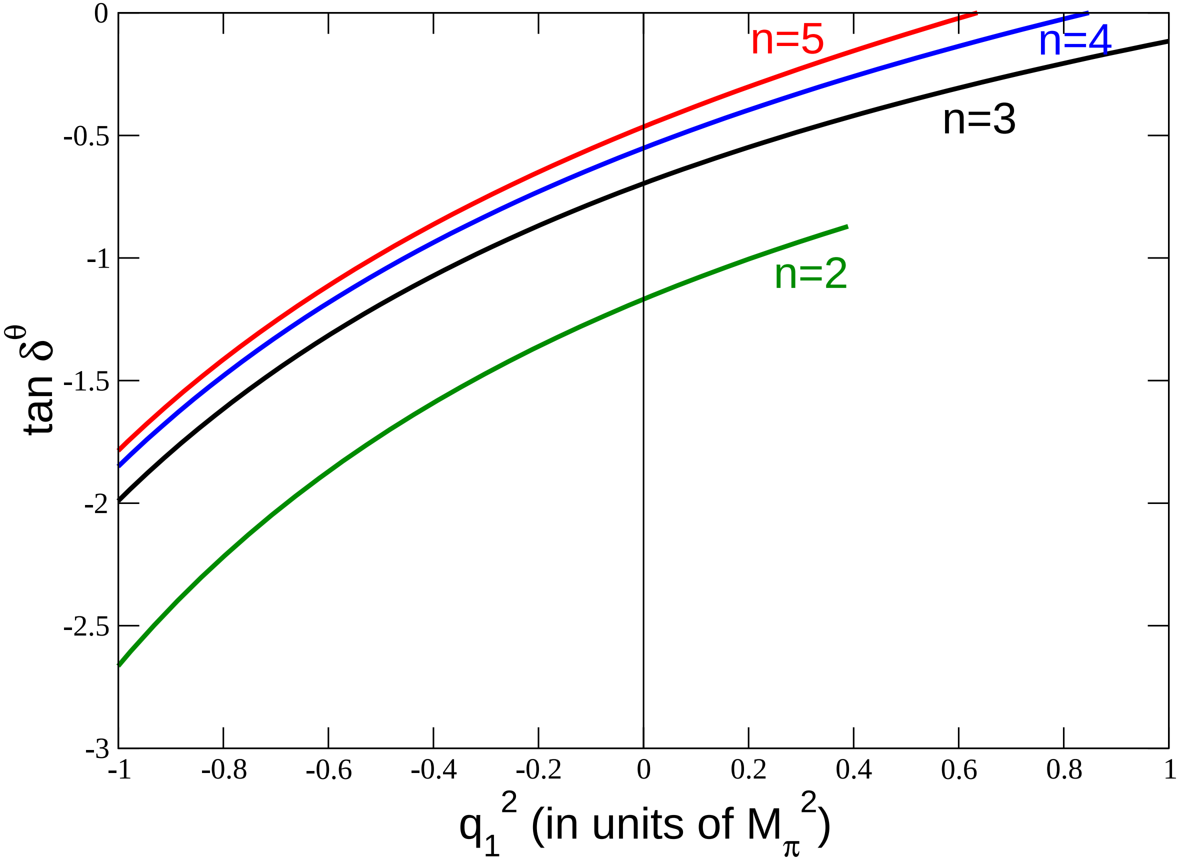
If the excited energy level does not change with θ, then $H_{12}(s_t) \simeq 0$

$\Rightarrow$ Corresponds to a tightly bound quark state (analogy with HBC)

Pseudophase $\pi\pi$, $K\bar{K}$ system: periodic b.c.



Pseudophase $\pi\pi$, $K\bar{K}$ system: anti-periodic b.c.



$q\bar{q}$ or tetraquarks?

Strangeness content in the non-relativistic quark model:

$$\mathbf{S}_i = \int d^3\mathbf{x} \; : \bar{\psi}_i(\mathbf{x},t)\psi_i(\mathbf{x},t) : , \qquad \mathbf{S}_i|\Phi\rangle = (N_i + \bar{N}_i)|\Phi\rangle$$

$$\Rightarrow \quad y_B = \frac{2\langle B|\bar{s}s|B\rangle}{\langle B|\bar{u}u + \bar{d}d|B\rangle}$$

B	$q\bar{q}$		$q^2\bar{q}^2$	
	wave funct.	y_B	wave funct.	y_B
$I=0$, ns	$\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}$,	0	$[ud][\bar{u}\bar{d}]$,	0
$I=0$, s	$s\bar{s}$,	∞	$\frac{[su][\bar{s}\bar{u}]+[sd][\bar{s}\bar{d}]}{\sqrt{2}}$,	2
$I=\frac{1}{2}$	$u\bar{s},\; d\bar{s}+$ conj.,	2	$[su][\bar{u}\bar{d}],\; [sd][\bar{u}\bar{d}]+$ conj.,	$\frac{2}{3}$
$I=1$	$u\bar{d},\; \frac{u\bar{u}-d\bar{d}}{\sqrt{2}}\; d\bar{u}$,	0	$[su][\bar{s}\bar{d}],\; \frac{[su][\bar{s}\bar{u}]-[sd][\bar{s}\bar{d}]}{\sqrt{2}},\; [sd][\bar{s}\bar{u}]$,	2

Feynman-Hellmann theorem:

$$y_B = 2\left(\frac{dM_B}{dm_s}\right)\cdot\left(\frac{dM_B}{dm_{ud}}\right)^{-1}$$

Example: matrix elements

1) External field method:

$$D_\mu = \partial_\mu + G_\mu + qA_\mu \quad \Rightarrow \quad U'_\mu = U_\mu U^B_\mu$$

Energy level shift in the constant magnetic field $A_y = Bx$, spin=1/2:

$$E_\pm = m \pm \mu B + O(B^2)\,, \quad \mu = g\,\frac{e}{2m}\,s \quad \Rightarrow \quad \mu = \frac{E_{1/2} - E_{-1/2}}{B}$$

2) Evaluation of the three-point function on the lattice:

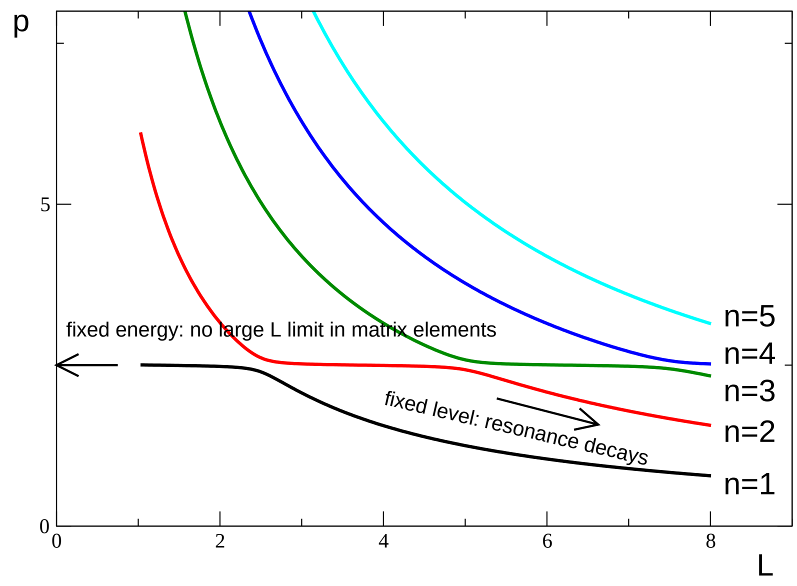
$$G^\alpha_{\mu\nu}(t,\tau,\mathbf{p}',\mathbf{p}) = \sum_{\mathbf{x},\xi} e^{-\mathbf{p}'(\mathbf{x}-\xi)-i\mathbf{p}\xi}\,\langle 0|\chi_\mu(x) J^\alpha(\xi)\chi_\nu(0)^\dagger|0\rangle$$

$$G_{\mu\nu}(t,\mathbf{p}) = \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}}\langle 0|\chi_\mu(x)\chi_\nu(0)^\dagger|0\rangle\,, \qquad \chi^\dagger_\mu = \bar{d}\gamma_\mu u$$

$$R^\alpha_{\mu\nu}(\tau,\mathbf{p}',\mathbf{p}) = \frac{G^\alpha_{\mu\nu}(t,\tau,\mathbf{p}',\mathbf{p})}{G_{\mu\mu}(t,\mathbf{p}')}\sqrt{\frac{G_{\nu\nu}(t-\tau,\mathbf{p})G_{\mu\mu}(\tau,\mathbf{p}')G_{\mu\mu}(t,\mathbf{p}')}{G_{\nu\nu}(\tau,\mathbf{p})G_{\mu\mu}(t-\tau,\mathbf{p}')G_{\nu\nu}(t,\mathbf{p})}} \rightarrow G_i(Q^2)$$

Infinite-volume limit of the matrix elements

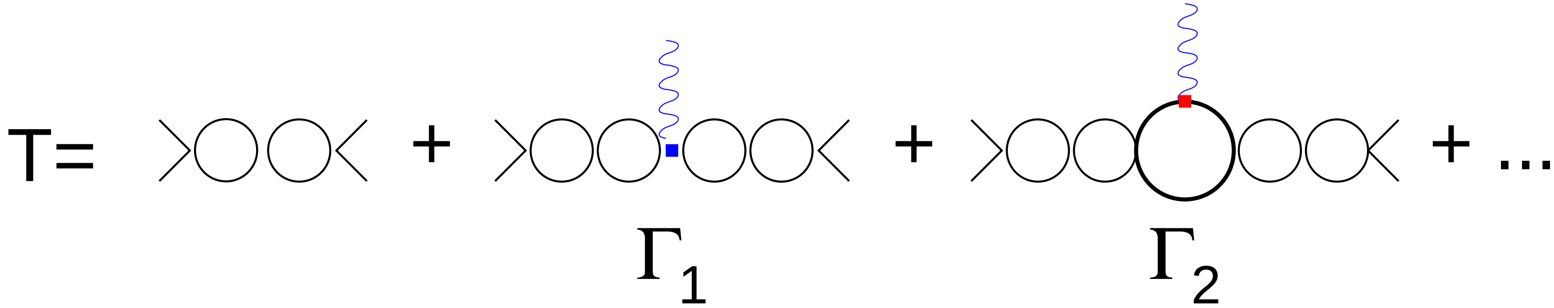
- For stable particles, the limit $L \to \infty$ exists
- Both methods give the matrix element sandwiched by the eigenvectors of the Hamiltonian. The resonances, however, do not correspond to a single energy level. How does one calculate the infinite-volume limit for these matrix elements?



- Fixed energy levels decay in the limit $L \to \infty$
- The matrix elements at fixed energy oscillate in the limit $L \to \infty$

Lüscher equation in the external field

Use <u>non-relativistic EFT</u> in a finite volume to calculate shift of the energy levels in the external field



$$H^{-1}(p,p) - J^L(p) - H^{-2}(p,p)\Gamma_1(p) - \Gamma_2(p) = 0$$

$$\hookrightarrow \quad \frac{2}{\sqrt{\pi}L} Z_{00}(1;q^2) = p\cot\delta(p) - 8\pi\sqrt{s}(H^{-2}(p,p)\Gamma_1(p) + \Gamma_2(p))$$

Determine shift of the pole position $\Rightarrow$ magnetic moment

Analytic continuation into the complex plane

$\Gamma_1(p)$: is a polynomial in p^2, can be analytically continued $p^2 \to p_*^2$

$$\Gamma_2(p) \sim \frac{dJ^L(p)}{dP_0} = \frac{W_N W_\pi - p^2}{8\pi ps}\,\cot\delta(p) + \frac{W_N W_\pi}{8\pi ps}\,\frac{q\phi'(q)}{\sin^2\phi(p)}$$

The solution of the problem:

- The "dangerous" term $\sim q\phi'(q)/\sin^2\phi(q)$ depends on the energy level index n through $q = q_n(p)$
- This term can be eliminated by measuring two energy levels $q_1(p)$ and $q_2(p)$

$$\hat{F}(p) = \frac{b_1 F_2 - b_2 F_1}{b_2 - b_1} \qquad \text{where} \quad F_i = \frac{Z_{00}(1, q_i^2)}{\pi^{3/2} q_i}\,, \quad b_i = q_i \phi'(q_i)$$

$\hookrightarrow$ The complex shift of the pole position is proportional to the magnetic moment of a resonance in the infinite volume

Results

- Non-relativistic effective field theories in a finite volume can be used to relate physical observables to the lattice data
- Using covariant NR EFT, Lüscher equation in the moving frame was derived in the non-equal mass case and for the scattering of the particles with spin
- Twisted boundary conditions are useful to study the multi-channel scattering:
  - $\Rightarrow$ *Extract both phase shift and inelasticity*
  - $\Rightarrow$ *Determine complex scattering lengths at inelastic threshold*
  - $\Rightarrow$ *Study molecular vs non-molecular nature of* $f_0(980), a_0(980)$
- Strangeness content can be used to establish the non-$q\bar{q}$ nature of the states
- Lüscher formula can be generalized for the resonance matrix elements (e.g., magnetic moments of $\Delta, \rho, \cdots$)