Functional RG for few-body systems

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Review of results from:

Schmidt and Moroz, arXiv:0910.4586
Krippa, Walet and Birse, arXiv:0911.4608
Krippa, Walet and Birse, arXiv:1011.5852

Background

Ideas of effective field theory and renormalisation group

- well-developed for few-nucleon and few-atom systems
- rely on separation of scales
- Wilsonian RG used to derive power counting

$\longrightarrow$ classify terms as perturbations around fixed point (or limit cycle)

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Many unsuccessful attempts to extend to dense matter

- but no separation of scales
- other EFT's for interacting Fermi systems exist
  (Landau Fermi liquid, Ginsburg-Landau theory)
- but parameters have no simple connection to underlying forces

EFTs based on contact interactions

- not well suited for standard many-body methods

$\longrightarrow$ switch to lattice simulation [Lee *et al*]
or look for some more heuristic approach

- based on field theory
- can be matched onto EFT's for few-body systems
(input from 2- and 3-body systems in vacuum)

Try functional renormalisation group ("exact" RG)

- based on Wilsonian RG approach to field theories
- successfully applied to various systems in for condensed-matter physics to quantum gravity
[version due to Wetterich (1993)]

Outline

- Functional RG
- Spin-$\frac{1}{2}$ fermions
- 4-body systems: dimer-dimer scattering
- Unitary limit: scaling
- Summary

Functional RG

Version based on the effective action $\Gamma[\phi_c]$

- start from generating function $W[J]$ defined by

$$e^{iW[J]} = \int D\phi\, e^{i(S[\phi]+J\cdot\phi-\frac{1}{2}\phi\cdot R\cdot\phi)}$$

- $R(q,k)$: regulator function
  suppresses modes with momenta $q \lesssim k$ ("cutoff scale")
- only modes with $q \gtrsim k$ integrated out
- $W[J]$ becomes full generating function as $k \to 0$

Legendre transform → effective action

$$\Gamma[\phi_c] = W[J] - J\cdot\phi_c + \frac{1}{2}\phi_c\cdot R\cdot\phi_c \quad \text{where} \quad \phi_c = \frac{\delta W}{\delta J}$$

(generating function for 1-particle-irreducible diagrams)

Γ evolves with scale k according to

$$\partial_k \Gamma = -\frac{i}{2} \operatorname{Tr}\left[(\partial_k R) \left(\Gamma^{(2)} - R \right)^{-1} \right] \quad \text{where} \quad \Gamma^{(2)} = \frac{\delta^2 \Gamma}{\delta\phi_c \delta\phi_c}$$

$(\Gamma^{(2)} - R)^{-1}$: propagator of boson in background field ϕ_c
(one-loop structure but still exact)

Evolution interpolates between "bare" classical action at large scale K and full 1PI effective action as $k \to 0$ (thresholds etc ...)

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Functional differential equation

- hard/impossible solve in general

$\longrightarrow$ work with tructated ansatz for Γ

- local action expanded in powers of derivatives
  (cf low-energy EFTs, but don't know *a priori* if we have a consistent power counting)

Derivative expansion may be good at starting scale K

- use power counting of EFT to determine relevant terms
  (or use this RG to find that power counting in scaling regime)
- but no guarantee that it remains good for $k \longrightarrow 0$
  (can't be for scattering amplitudes at energies above threshold:
  cuts $\longrightarrow$ nonanalytic behaviour)

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$\longrightarrow$ need consistency checks:
stability against adding extra terms to ansatz
stability against changes in form of regulator

- use this to optimise choice of regulator [Litim, Pawlowski]

Two species of fermion

Fermion field: $\psi(x)$ (spin-$\frac{1}{2}$ atoms or neutrons)
Boson “dimer” field: $\phi(x)$ (strongly interacting pairs)
Local (nonrelativistic) ansatz for action in vacuum: 2-body sector

$$
\begin{aligned}
&\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] \\
&= \int \mathrm{d}^4x \left[\psi(x)^\dagger \left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{2M}\right)\psi(x)\right. \\
&\qquad\qquad +Z_\phi(k)\,\phi(x)^\dagger \left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{4M}\right)\phi(x) - u_1(k)\,\phi(x)^\dagger\phi(x) \\
&\qquad\qquad \left. -g\left(\frac{\mathrm{i}}{2}\,\phi(x)^\dagger\psi(x)^{\mathrm{T}}\sigma_2\psi(x) + \mathrm{H\,c}\right)\right]
\end{aligned}
$$

g: AA$\longrightarrow$D coupling
$u_1(k)$: dimer self-energy (u_1/g^2: only physical parameter)
$Z_\phi(k)$: dimer wave-function renormalisation

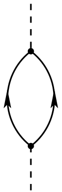
Evolution equation

$$\partial_k \Gamma = +\frac{\mathrm{i}}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_F)\left((\mathbf{\Gamma}^{(2)}-\mathbf{R})^{-1}\right)_{FF}\right] - \frac{\mathrm{i}}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_B)\left((\mathbf{\Gamma}^{(2)}-\mathbf{R})^{-1}\right)_{BB}\right]$$

$\mathbf{\Gamma}^{(2)}$: matrix of second derivatives of the action

(Gorkov-like form: ψ and $\psi^\dagger$ as independent variables $\longrightarrow$ factors of $\frac{1}{2}$)

"Skeleton" diagram for driving terms in evolution of 2-body parameters



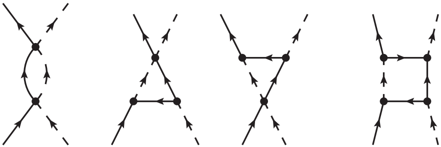
(need to insert $\partial_k \mathbf{R}_F$ on one internal line)

Expand in powers of energy $\longrightarrow \partial_k u_1, \partial_k Z_\phi$

3-body sector: AD contact interaction

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] = \cdots - \lambda(k)\int d^4x\,\psi^\dagger(x)\phi^\dagger(x)\phi(x)\psi(x)$$

Evolution of λ driven by terms corresponding to skeletons



- AD contact interaction
- single-A exchange between dimers
  (cf Faddeev and STM equations)

4-body sector: DD$\longrightarrow$DD, DD$\longrightarrow$DAA, DAA$\longrightarrow$DAA terms

[cf Schmidt and Moroz (2009): bosonic case]

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] = \cdots - \int d^4x \left[\frac{1}{2}u_2(k)\left(\phi^\dagger\phi\right)^2 + \frac{1}{4}v(k)\left(\phi^{\dagger 2}\phi\,\psi^{\mathrm{T}}\psi + \mathrm{H\,c}\right) + \frac{1}{4}w(k)\,\phi^\dagger\phi\,\psi^\dagger\psi^{\dagger\mathrm{T}}\psi^{\mathrm{T}}\psi\right]$$

- dimer "breakup" terms allow 3-body physics to feed in properly (cf Faddeev-Yakubovski)

$\longrightarrow$ coupled evolution equations for u_2, v, w (27 distinct skeletons)

Regulators

- fermions: sharp cutoff

$$R_F(\boldsymbol{q}, k) = \frac{k^2 - q^2}{2M}\,\theta(k - q)$$

- pushes states with $q > k$ up to energy $k^2/2M$
- nonrelativistic version of "optimised" cutoff [Litim (2001)]
- fastest convergence at this level of truncation
- bosons

$$R_B(\boldsymbol{q}, k) = Z_\phi(k)\,\frac{(c_B\,k)^2 - q^2}{4M}\,\theta(c_B\,k - q)$$

- c_B: relative scale of boson cutoff
- optimised choice $c_B = 1$ [cf Pawlowski (2007)]
  (no mismatch between fermion and boson cutoffs)

Also examined smooth cutoffs – more convenient in dense matter

Initial conditions

As $k \longrightarrow \infty$ boson field purely auxiliary

- $Z_\phi(k) \longrightarrow 0$
- $u_1(K)$ chosen so that in physical limit ($k \longrightarrow 0$)

$$u_1(0) = -\frac{M g^2}{4\pi a_F} \qquad a_F\text{: AA scattering length}$$

- other couplings λ, u_2, v, w also vanish as $k \longrightarrow \infty$

$\longrightarrow$ either set $Z_\phi(K) = 0$ etc at large starting scale K
or match on to K^{-n} behaviour in scaling regime $K \gg 1/a_F$

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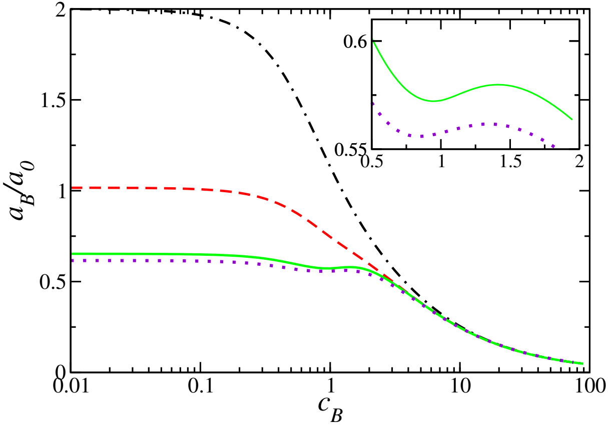
Expansion point: dimer binding energy $\mathcal{E}_D = -1/(M\,a_F^2)$

- external boson lines carry $P_0 = -\mathcal{E}_D$
- external fermion lines carry $P_0 = -\mathcal{E}_D/2$
  (below all thresholds)

Results: DD scattering length



- black: "minimal" action – only two-body and DD vertex u_2
- red adds three-body coupling λ
- green: full local four-body action, includes v, w
- purple: similar but using smooth cutoff

Comments

- results seem to converge as more terms are included
- converge to value only weakly dependent on cutoff
  (very liitle variation over range $0 \leq c_B \lesssim 2$)
- stationary very close to expected "optimum" $c_B = 1$
- incomplete actions $\longrightarrow$ strong dependence on c_B around $c_B = 1$

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Final result

- $a_B/a_F \simeq 0.58 \pm 0.02$
- agrees well with full few-body result $a_B/a_F = 0.6$
  [Petrov, Salomon and Shlyapnikov (2004)]

Unitary limit

Tune $u_1(K)$ so that $u_1(k) \to 0$ as $k \to 0$ $(1/a_F = 0)$
Evolution equation for 3-body coupling λ

$$\partial_k \lambda = \frac{28\,k}{125\,g^2\,M}\lambda^2 + \frac{156}{125\,k}\lambda + \frac{128\,g^2\,M}{125\,k^3}$$

Rescale: $\widehat{\lambda} = \dfrac{k^2}{g^2\,M}\lambda$

- dimensionless equation

$$k\partial_k \widehat{\lambda} = \frac{28}{125}\widehat{\lambda}^2 + \frac{406}{125}\widehat{\lambda} + \frac{128}{125}$$

$\longrightarrow$ two fixed point solutions (roots of RHS)

- expand around IR stable point: $\widehat{\lambda} - \widehat{\lambda}_0 \propto k^\nu$ with $\nu = 3.10355$
- compare exact solution: $\nu = 4.33244$

  [Griesshammer (2005); Werner and Castin; Birse (2006)]

Bosons (or ≥ 3 species of fermion in symmetric channel)

Very similar action and evolution equations

- different numerical coefficients
  $\partial_k\lambda$ term linear in λ gets factor of -2 (cf Faddeev equation)
- rescaled equation

$$k\partial_k\widehat{\lambda} = \frac{56}{125}\widehat{\lambda}^2 - \frac{62}{125}\widehat{\lambda} + \frac{256}{125}$$

$\longrightarrow$ two complex roots – fixed points

- expand around either: $\widehat{\lambda} - \widehat{\lambda}_0 \propto k^{\pm 2\mathrm{i}s_0}$
- imaginary exponent $\rightarrow$ limit cycle of Efimov effect
- real solutions periodic under scaling k by factor e^{π/s_0}
  where $s_0 = 0.92503$ [Schmidt and Moroz (2009)]
- agrees with Efimov $s_0 = 1.00624$ to $\sim 5\%$

4-body systems

Rescaled evolution equations for u_2, v, w

Fermions

- 4 fixed-point solutions
- only one IR stable
- smallest eigenvalue $\nu = 4.19149$ (irrelevant)
- compare with result from system in harmonic trap $\nu = 5.0184$ [Stecher and Greene (2009)]

Bosons

- 4 complex fixed points (since λ complex)
- only one IR stable
- eigenvalue with smallest real part $\nu = 0.055165 + 3.50440\mathrm{i}$

$\longrightarrow$ very weakly irrelevant ??

- couplings flow to cycle driven by 3-body sector $\lambda(k)$
- no sign of 4-body bound states at this truncation

Summary

First full applications of functional RG to 3- and 4-body systems

- local truncation, “optimised” cutoff

$\longrightarrow$ results for dimer-dimer scattering length
stable against variation of cutoff
agree with direct few-body calculations

- unitary limit: scaling behaviours agree with exact 3-body
qualitatively for 2 species of fermion
much more accurately for bosons
- estimates of anomalous dimensions for four-body forces

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Future work

- use these 3-, 4-body interactions as input into calculations of dense matter [Floerchinger, talk at this meeting]
- 4 species of fermion (nucleons)
  SU(4) symmetry: evolution same as either bosons or 2 species

3-body physics in unitary limit

Momentum space: one-variable integral equation

[Skornyakov and Ter-Martirosian (1956)]

Faddeev equation in hyperspherical coordinates

$(R^2 = |\mathbf{r}_1 - \mathbf{r}_2|^2 + |\mathbf{r}_2 - \mathbf{r}_3|^2 + |\mathbf{r}_3 - \mathbf{r}_1|^2)$

- Schrödinger equation with $1/R^2$ potential [Efimov, 1971]

$$-\frac{1}{M}\left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R} - \frac{\nu^2}{R^2}\right] u(r) = p^2\, u(R)$$

- hyperangular eigenvalue ν^2 fixed by boundary condition (S-waves)

$$1 = \sigma \frac{4}{\sqrt{3\pi}\,\nu} \frac{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \sin\left(\frac{\pi\nu}{6}\right)$$

- spatially symmetric: $\sigma = +1$; mixed-symmetry $\sigma = -\frac{1}{2}$ (“particle-exchange interaction” between pair and third particle)