The hadronic vacuum polarisation contribution to $(g-2)_\mu$ from lattice QCD

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In collaboration with:

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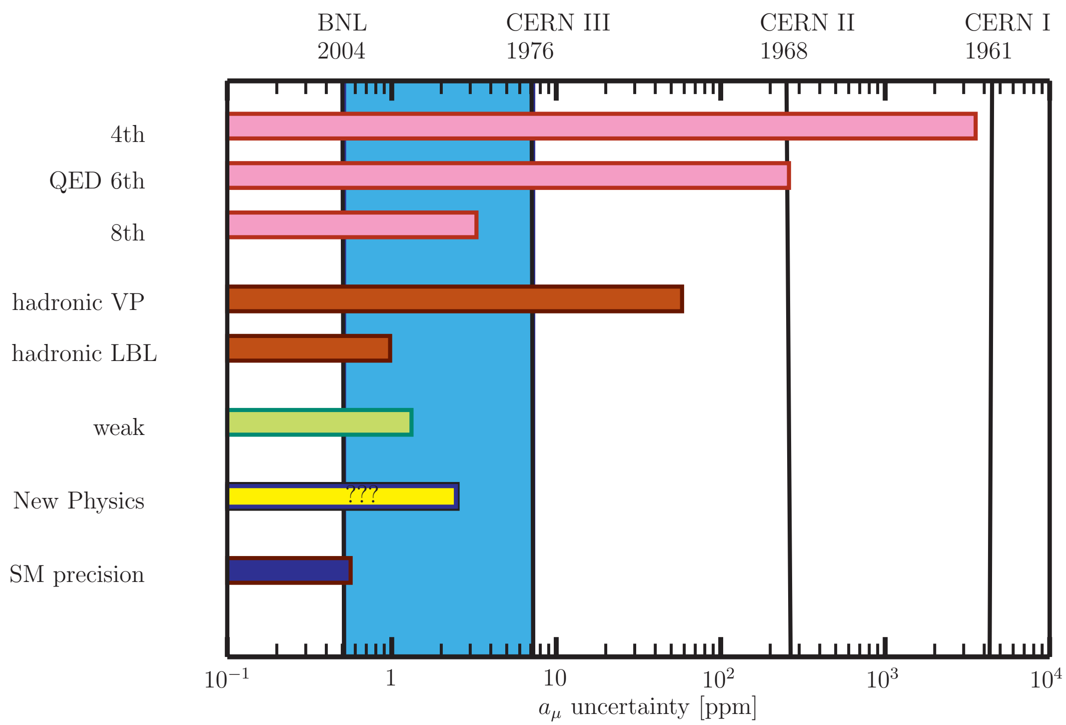
Introduction

- Muon anomalous magnetic moment: $a_\mu = \frac{1}{2}(g-2)_\mu$

$$a_\mu = \begin{cases} 116\,592\,080(63) \cdot 10^{-11} & \text{Experiment} \\ 116\,591\,790(65) \cdot 10^{-10} & \text{SM prediction}^{*} \end{cases} \qquad (3.2\sigma \text{ tension})$$

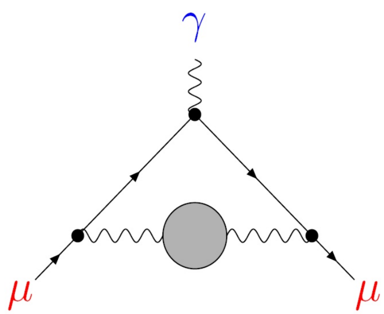
- Experimental sensitivity versus individual contributions:



[Jegerlehner & Nyffeler, Phys Rept 477 (2009) 1]*

Hadronic vacuum polarisation

- Leading contribution:



- Phenomenological approach:

$$a_\mu^{\mathrm{VP;had}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m_\pi^2}^{E_{\mathrm{cut}}^2} \mathrm{d}s \frac{R_{\mathrm{had}}^{\mathrm{data}}(s)\hat{K}(s)}{s^2} + \int_{E_{\mathrm{cut}}^2}^{\infty} \mathrm{d}s \frac{R_{\mathrm{had}}^{\mathrm{pQCD}}(s)\hat{K}(s)}{s^2} \right\}$$

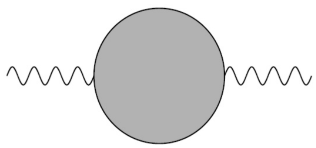
$$\Rightarrow \qquad a_\mu^{\mathrm{VP;had}} = \begin{cases} (609.75 \pm 4.72) \cdot 10^{-10} & \text{(combined } e^+e^-\text{-data)} \\ (609.96 \pm 4.65) \cdot 10^{-10} & (e^+e^- \text{ and } \tau\text{-data}) \end{cases}$$

- After accounting for $\rho - \gamma$ mixing the 3σ-tension persists

[Jegerlehner & Szafron, arXiv:1101.2872]

Lattice approach to hadronic vacuum polarisation

- Euclidean vacuum polarisation tensor:



$$\Pi_{\mu\nu}(q) = i \int \mathrm{d}^4x \, \mathrm{e}^{iq\cdot(x-y)} \, \langle J_\mu(x) J_\nu(y) \rangle \equiv (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2)$$

$$J_\mu(x) = \sum_{q=u,d,s,\ldots} Q_q \, \overline{q}(x) \gamma_\mu q(x)$$

- Determine a_μ^{had} from convolution integral:

$$a_\mu^{\mathsf{had}} = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}q^2 \, f(q^2) \{\Pi(q^2) - \Pi(0)\}$$

$$f(q^2) = \frac{m_\mu^2 q^2 Z^3 (1 - q^2 Z)}{1 + m_\mu^2 q^2 Z^2}, \qquad Z = -\frac{q^2 - \sqrt{q^4 + 4m_\mu^2 q^2}}{2m_\mu^2 q^2}$$

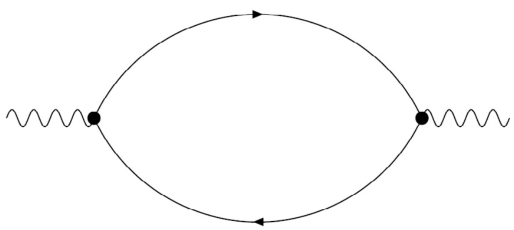
Problems for lattice calculations:

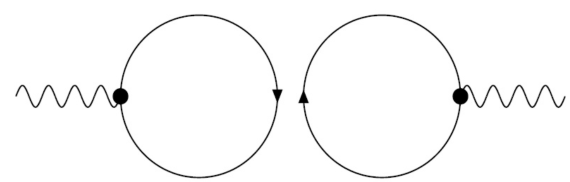
- Convolution integral dominated by momenta near m_μ :

  maximum of $f(q^2)$ located at: $(\sqrt{5}-2)m_\mu^2 \approx 0.003\,\mathrm{GeV}^2$

  lowest momentum transfer: $\left(\frac{2\pi}{T}\right)^2 \approx 0.06\,\mathrm{GeV}^2$

- Contributions from quark disconnected diagrams





Large noise-to-signal ratio

- Contributions from vector resonances $(\rho,\ \omega,\ \phi)$ must be included

Outline:

1. ChPT and the rôle of disconnected diagrams

2. Lattice set-up

3. Results

4. Summary

2. ChPT and the rôle of disconnected diagrams

Aim:

- Show that connected and disconnected contributions have separate continuum and finite-volume limits

- Compute the relative size of the disconnected contribution in ChPT

Connected & disconnected diagrams in Partially Quenched QCD

[Della Morte & Jüttner, JHEP 11 (2010) 154]

- Consider two-flavour QCD:

$$\mathcal{L}_{\text{QCD}}^{\text{quark}} = \overline{u}(\not{D} + m_u)u + \overline{d}(\not{D} + m_d)d$$

- Consider contribution from up-quark only:

$$\Pi_{\mu\nu}^{uu}(q) = i\frac{4}{9}\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\, \left\langle j_\mu^{uu}(x) j_\nu^{uu}(0)\right\rangle_{\text{QCD}}$$

$\rightarrow$ Wick contractions yield connected and disconnected parts

- The same result is recovered in partially quenched QCD:

  Add a mass-degenerate valence quark r and a ghost field r_g:

$$\mathcal{L}_{\text{PQQCD}}^{\text{quark}} = \overline{u}(\not{D} + m_u)u + \overline{d}(\not{D} + m_d)d + \overline{r}(\not{D} + m_u)r + r_g^\dagger(\not{D} + m_u)r_g$$

- Partition functions of QCD and PQQCD are mathematically the same

  PQQCD: based on extended graded flavour symmetry group

- Rewrite contribution from up-quark:

$$\Pi_{\mu\nu}^{uu}(q) = i\frac{4}{9}\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\, \langle j_\mu^{uu}(x) j_\nu^{uu}(0)\rangle_{\mathrm{QCD}}$$
$$= i\frac{4}{9}\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\left\{\langle j_\mu^{ur}(x) j_\nu^{ru}(0)\rangle_{\mathrm{PQQCD}} + \langle j_\mu^{uu}(x) j_\nu^{rr}(0)\rangle_{\mathrm{PQQCD}}\right\}$$

$\rightarrow$ Connected and disconnected contributions are expressed as separate correlation functions in PQQCD

- Contribution from down-quark treated in the same way $(m_u = m_d)$

- Low-energy description: Partially Quenched Chiral Perturbation Theory

QCD	PQQCD, PQChPT
$N_\mathrm{f} = 2$	$\mathrm{SU}(3\|1)$
$N_\mathrm{f} = 2$, quenched strange	$\mathrm{SU}(4\|2)$
$N_\mathrm{f} = 3$	$\mathrm{SU}(4\|1)$

Connected & disconnected diagrams in Partially Quenched ChPT

[Della Morte & Jüttner, JHEP 11 (2010) 154]

- Leading-order Lagrangian:

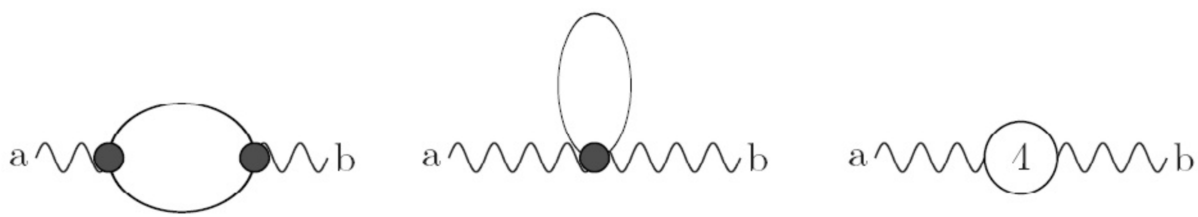
$$\mathcal{L}^{(2)} = \frac{F_0}{4}\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{1}{2}\mathrm{Tr}\left(MU^\dagger + M^\dagger U\right), \quad U = \exp(i\lambda_a\phi_a F_0)$$

$$D_\mu U = \partial_\mu U + iv_\mu U - iUv_\mu$$

$$\mathcal{L}^{(2)}_{\mathrm{int}} = -f_{abc}\phi_c\partial_\mu\phi_a v^b_\mu + \tfrac{1}{2}f_{abg}f_{cdg}\phi_b\phi_c v^a_\mu v^d_\mu$$

- Graded flavour symmetry: Tr $\rightarrow$ Str

- One-loop contributions to vector-vector correlator:



- Remove divergencies by tree-level insertions of $\mathrm{O}(p^4)$ Lagrangian
  [R. Kaiser, Phys Rev D63 (2001) 076010]

$N_\mathrm{f} = 2$ **QCD in SU$(3|1)$ PQChPT**

- Result of one-loop calculation:

$$\Pi^{(3|1)}(q^2) = - \left(\Lambda^{(3|1)}(\mu) + \frac{2}{9} h_s + 4i\bar{B}_{21}(\mu^2, q^2, m_\pi^2)\right)$$

$$\Pi^{(3|1)}_{\text{conn}}(q^2) = -\frac{10}{9} \left(\Lambda^{(3|1)}(\mu) + 4i\bar{B}_{21}(\mu^2, q^2, m_\pi^2)\right)$$

$$\Pi^{(3|1)}_{\text{disc}}(q^2) = \frac{1}{9} \left(\Lambda^{(3|1)}(\mu) - 2h_s + 4i\bar{B}_{21}(\mu^2, q^2, m_\pi^2)\right)$$

- Low-energy constants: $\Lambda^{(3|1)}(\mu) = -8h_2(\mu),\ h_s$

- Combination $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$ enters convolution integral

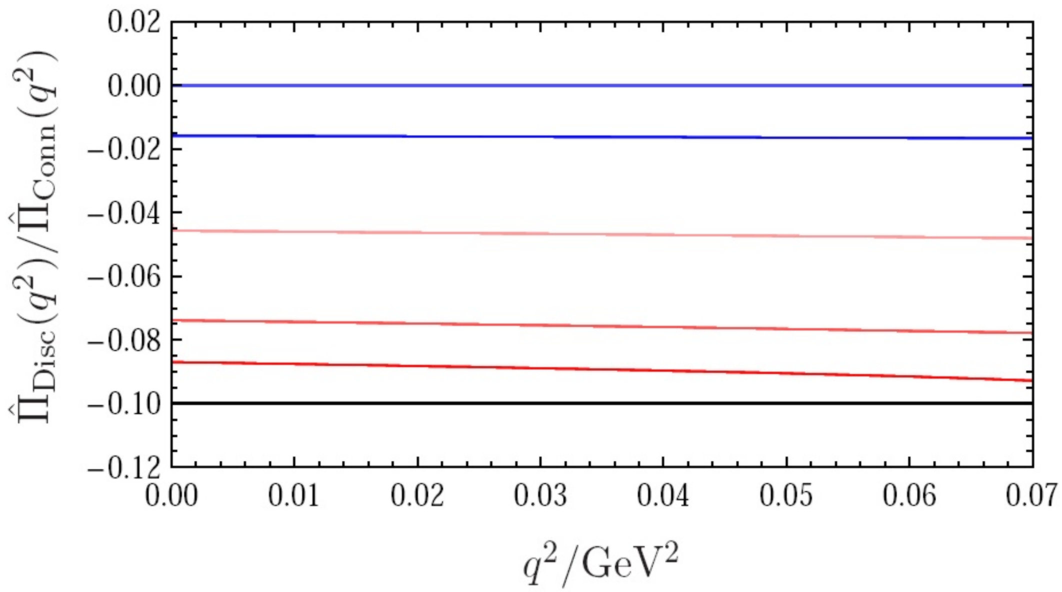
$$\Rightarrow \quad \frac{\Pi_{\text{disc}}(q^2) - \Pi_{\text{disc}}(0)}{\Pi_{\text{conn}}(q^2) - \Pi_{\text{conn}}(0)} = -\frac{1}{10}$$

$\Rightarrow$ PQChPT @ NLO: disconnected contribution is 10% downward shift

$N_f = 2$ QCD with quenched strange quark, in SU$(4|2)$ PQChPT

- Disconnected contribution vanishes for $m_\pi = m_K$

  $\rightarrow$ study contribution as a function of $m_\pi = 495\,\text{MeV}, \ldots, 139\,\text{MeV}$



-10% correction of two-flavour case presents lower bound

$N_{\rm f} = 3$ **QCD in SU**$(4|1)$ **PQChPT**

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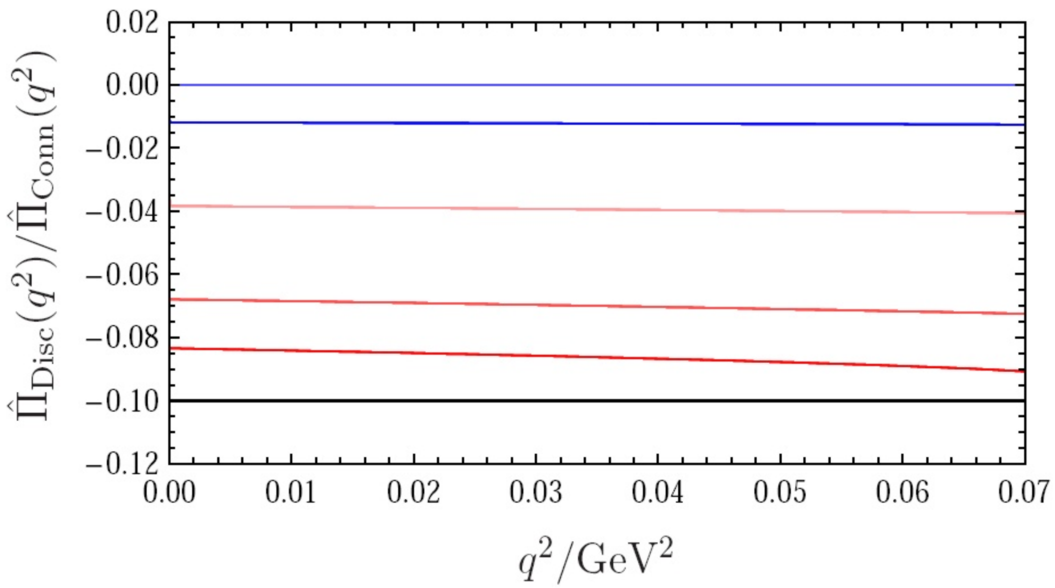
$\rightarrow$ study contribution as a function of $m_\pi = 495\,{\rm MeV}, \ldots, 139\,{\rm MeV}$



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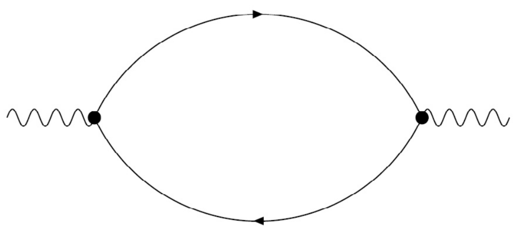
2. Lattice setup

- Focus on connected contribution:

$$\Pi(q^2) = \frac{i \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\, \langle J_\mu^{rs}(x) J_\nu^{sr}(0)\rangle}{q_\mu q_\nu - g_{\mu\nu} q^2}$$



- $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$ enters convolution integral

$\rightarrow$ requires extrapolation to $q^2 = 0$

- Lattice momenta: $q_\mu = n_\mu \frac{2\pi}{L_\mu}, \quad n_\mu = 0, 1, \ldots, L_\mu/a - 1$

$$L = 2.5\,\mathrm{fm}, \quad T = 2L \quad \Rightarrow \quad q^2 \gtrsim 0.06\,\mathrm{GeV}^2$$

$\rightarrow$ Lack of accurate data points near $q^2 = 0$

$\rightarrow$ Extrapolation to $q^2 = 0$ not well controlled

Twisted boundary conditions

[Bedaque 2004; de Divitiis, Petronzio & Tantalo 2004; Flynn, Jüttner & Sachrajda 2005]

- Apply "twisted" spatial boundary conditions;

  Impose periodicity up to a phase $\vec{\theta}$:

$$\psi(x + L\hat{e}_k) = \mathrm{e}^{i\theta_k}\psi(x) \quad \Rightarrow \quad q_k = n_k\frac{2\pi}{L} + \frac{\theta_k}{L}$$

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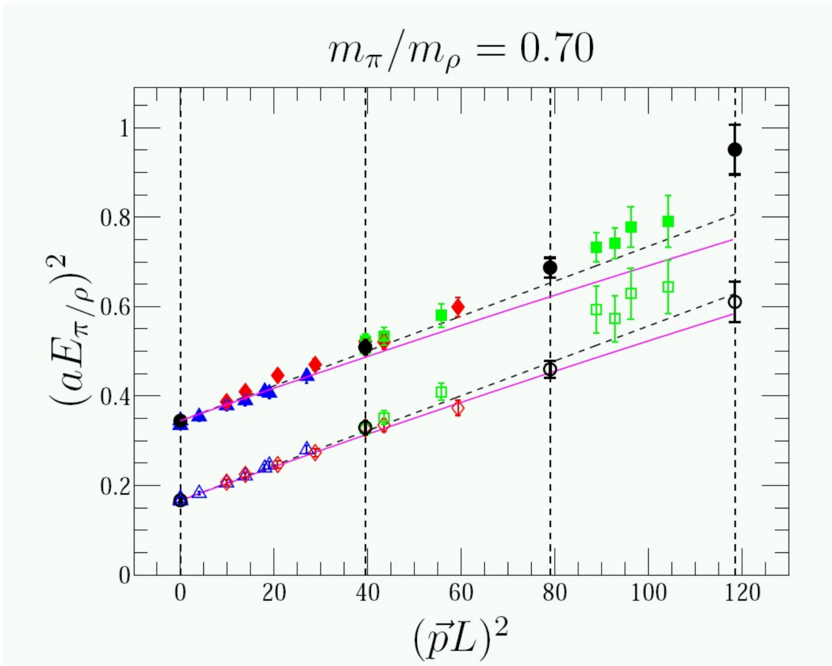
- Check dispersion relation:



[Flynn, Jüttner, Sachrajda, hep-lat/0506016]

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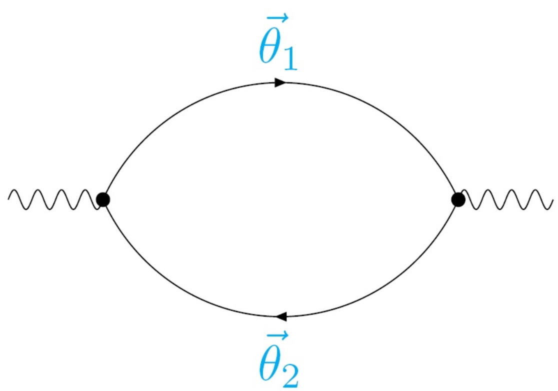
- Imposing twisted boundary conditions in valence sector only:

  exponentially small finite-volume effects *[Sachrajda & Villadoro, Phys Lett B609 (2005) 73]*

- Can tune q^2 to any desired value

  $\rightarrow$ Compute connected contribution to $\Pi(q^2)$



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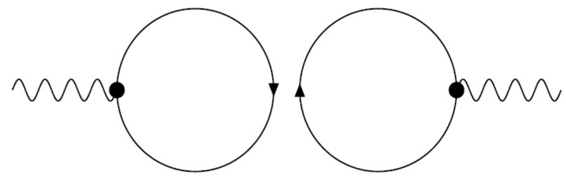
- Effect of twist angle cancels in disconnected contribution to $\Pi(q^2)$

  $\rightarrow$ Compute disconnected diagrams for Fourier modes only;

  $\rightarrow$ Validate their relative suppression



CLS† Run Table

- Discretisation: $N_\mathrm{f} = 2$ flavours of $\mathrm{O}(a)$ improved Wilson quarks
- 3 lattice spacings: $a = 0.08,\ 0.066,\ 0.053\,\mathrm{fm}$
- Pion masses: $m_\pi = 250 - 700\,\mathrm{MeV}$

β	$a[\mathrm{fm}]$	lattice	$L[\mathrm{fm}]$	masses	$m_\pi L$	Labels
5.20	0.08	$32^3 \cdot 64$	2.6	4 masses	4.7 – 7.9	A2 — A5
5.20	0.08	$48^3 \cdot 96$	3.8	1 mass	5.4	B6
5.30	0.07	$32^3 \cdot 64$	2.2	3 masses	4.7 – 7.9	E3 — E5
5.30	0.07	$48^3 \cdot 96$	3.4	2 masses	5.0, 4.2	F6, F7
5.50	0.05	$48^3 \cdot 96$	2.5	3 masses	5.3 – 7.7	N3 — N5
5.50	0.05	$64^3 \cdot 128$	3.4	1 mass	4.2	O7

[Brandt, Capitani, Della Morte, Djukanovic, von Hippel, Jäger, Jüttner, Knippschild, H.W., arXiv:1010.2390]

†CLS = **C**oordinated **L**attice **S**imulations

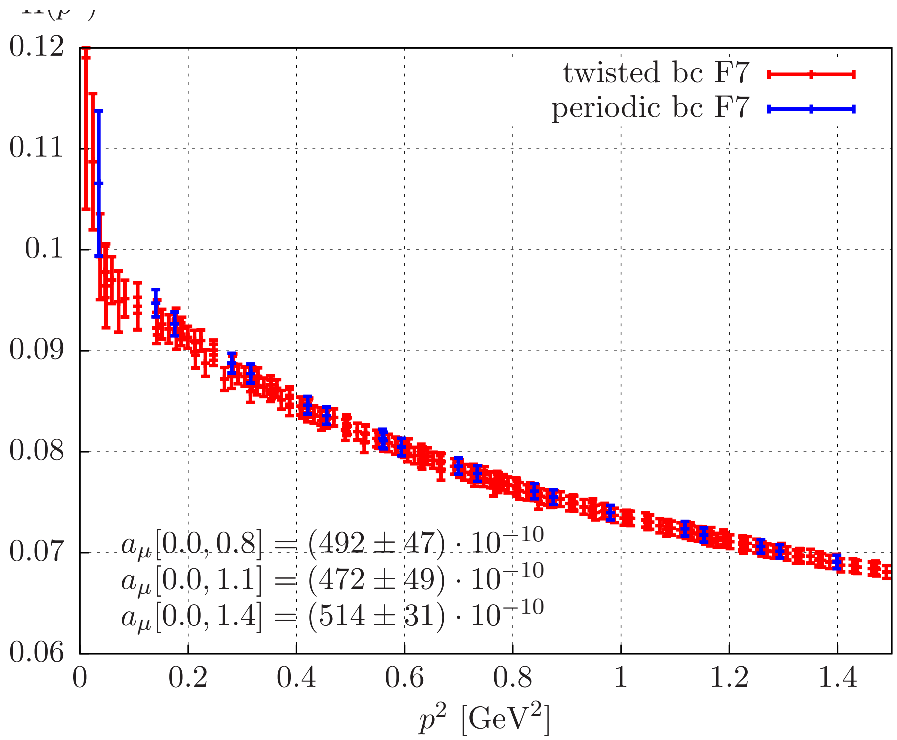
[https://twiki.cern.ch/twiki/bin/view/CLS/WebHome]

3. Results

[Della Morte, Jäger, Jüttner, H.W., arXiv:1011.5793, and in preparation]

- Compute the connected contribution using twisted boundary conditions
- Use the conserved point-split lattice vector current
- Results in pure $N_{\rm f} = 2$ QCD and in two-flavour QCD with a quenched strange quark
- Investigate systematic effects: lattice artefacts, finite-volume effects

- $\Pi(q^2)$ at $m_\pi \approx 250\,\mathrm{MeV}, a = 0.066\,\mathrm{fm}$



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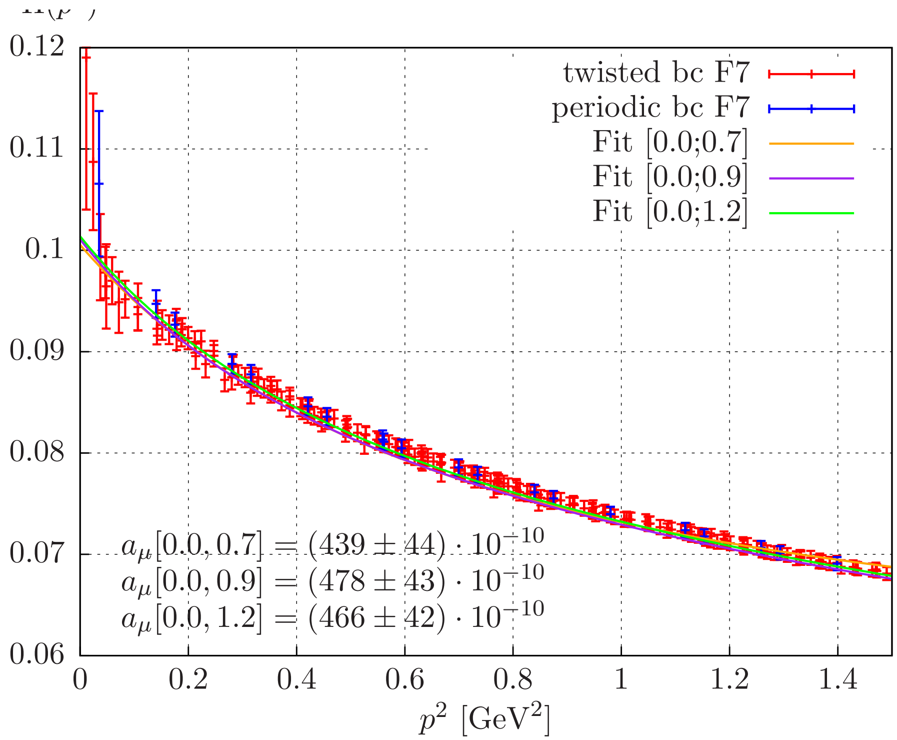
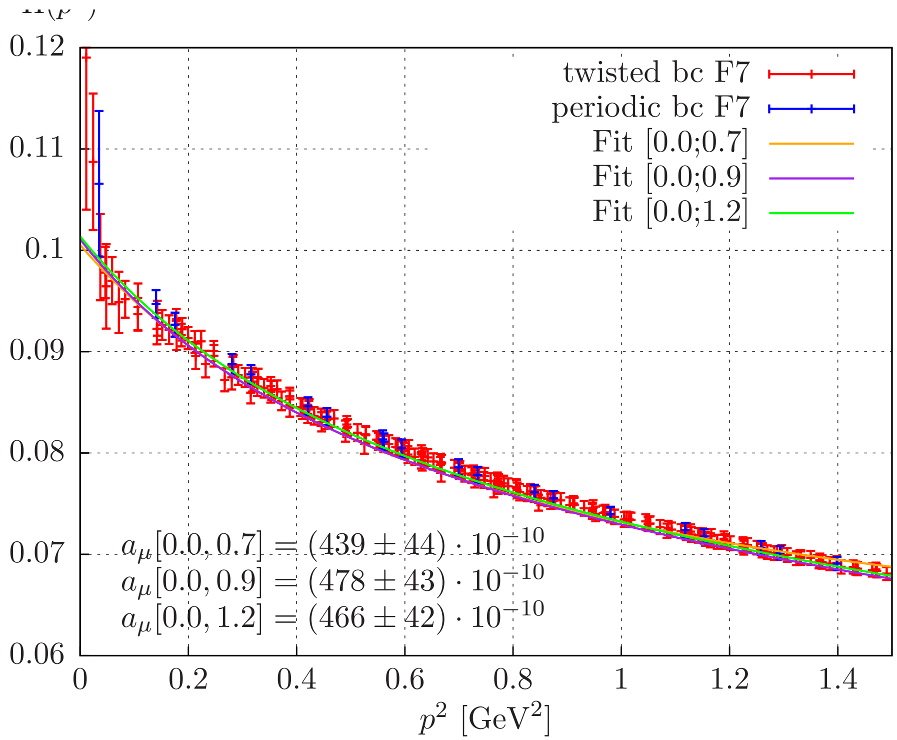


- Use different *ansätze* to determine $\Pi(0)$:
  - Polynomial
  - Dispersion relation: $\Pi(q^2) = B \ln(a^2 q^2 + a^2 s_0) - \frac{A}{q^2 + m_V^2} + K$
  - Padé fit

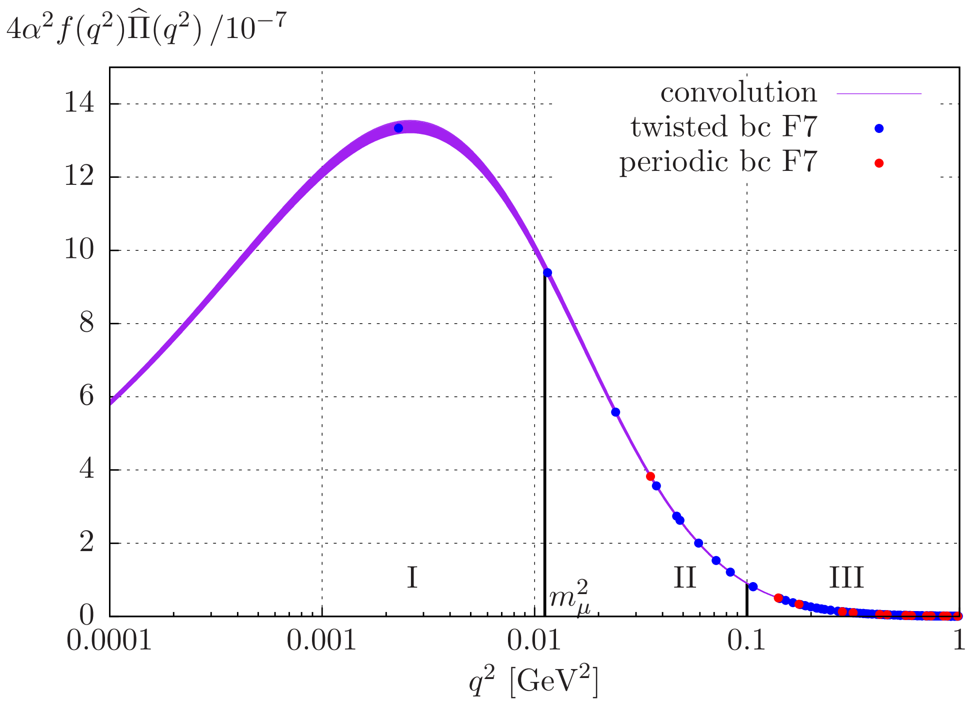
- $\Pi(q^2)$ at $m_\pi \approx 250\,\mathrm{MeV}, a = 0.066\,\mathrm{fm}$



- Twisted boundary conditions stabilise determination of q^2-dependence and of $\Pi(0)$:

  Systematic uncertainties arising from ambiguity in the fit ansatz and choice of q^2-range greatly reduced

- Convolution integral for $m_\pi \approx 250\,\mathrm{MeV}, a = 0.066\,\mathrm{fm}$



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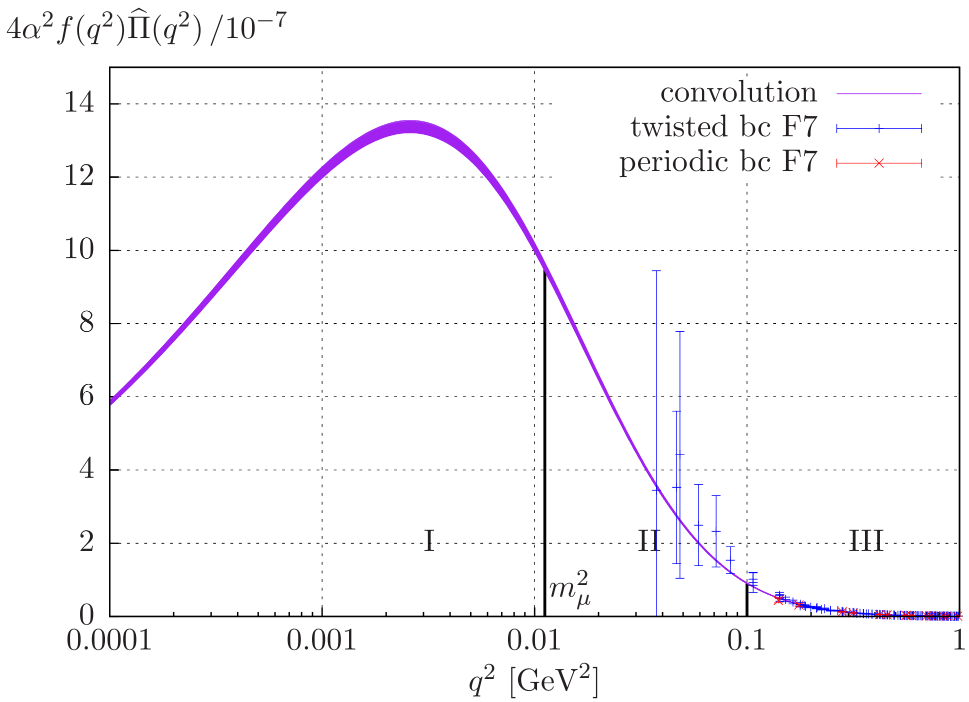

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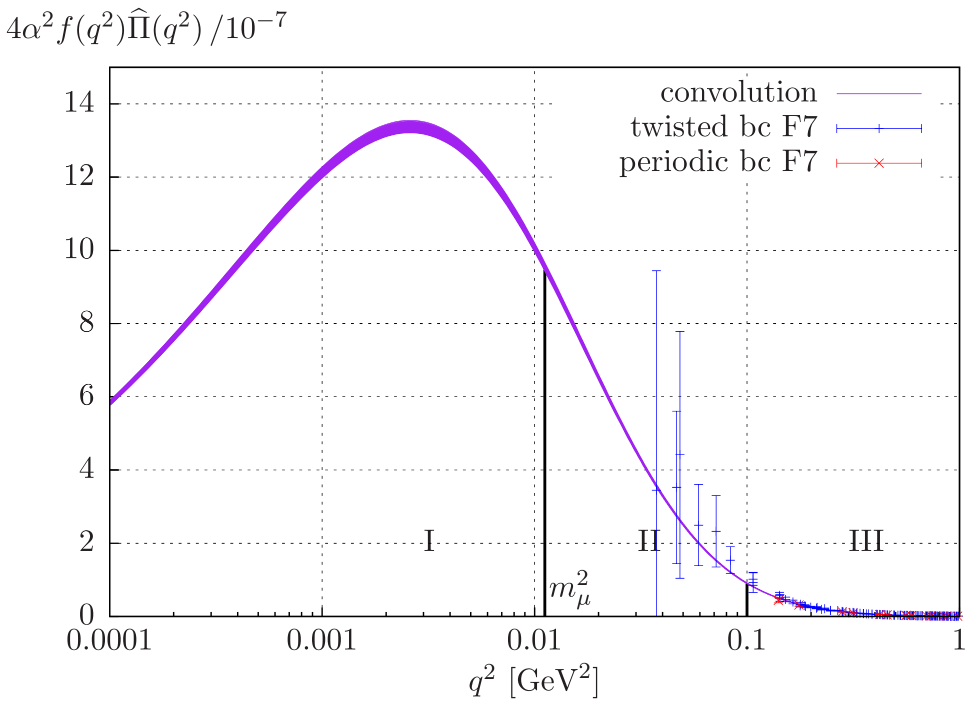

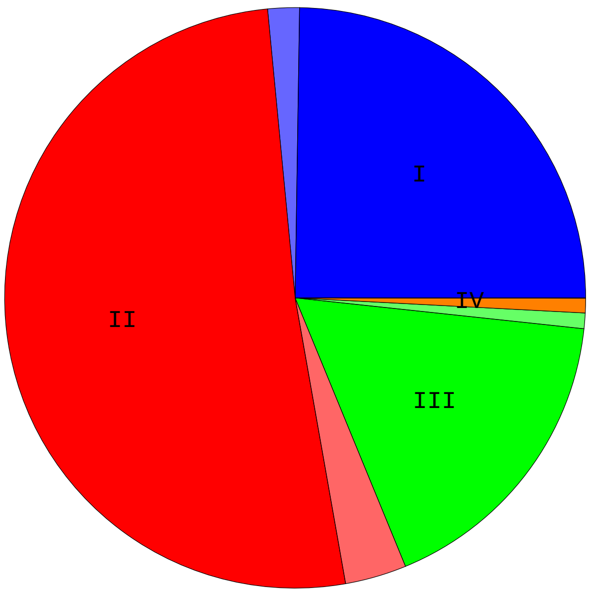

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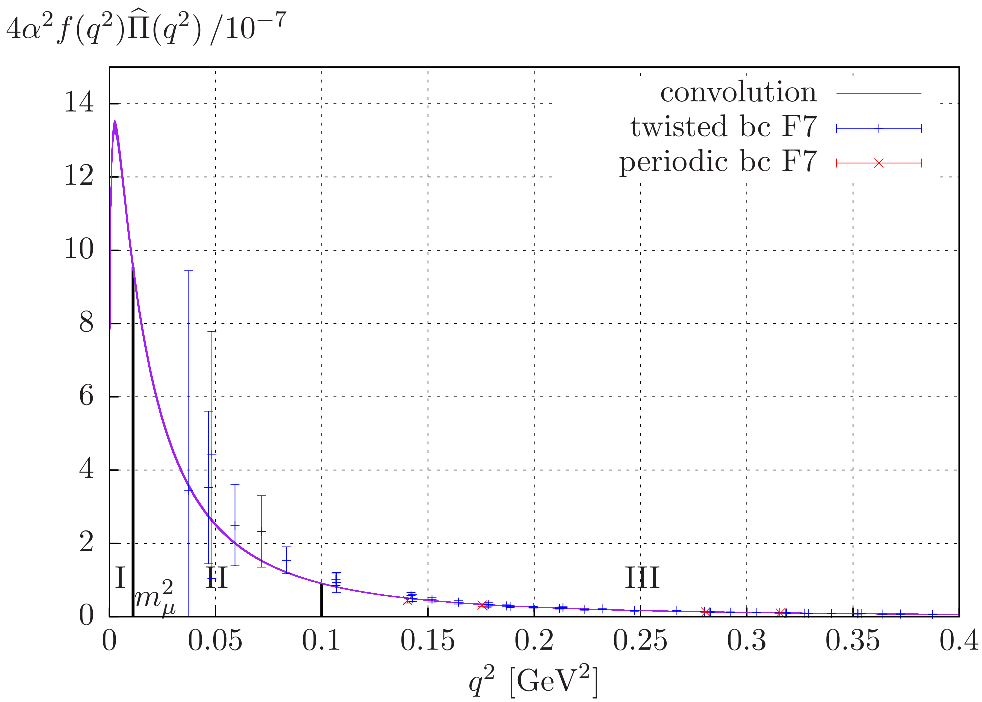
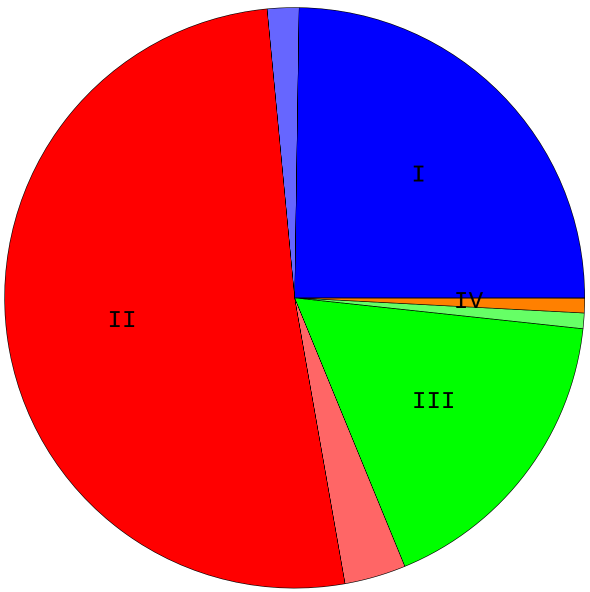
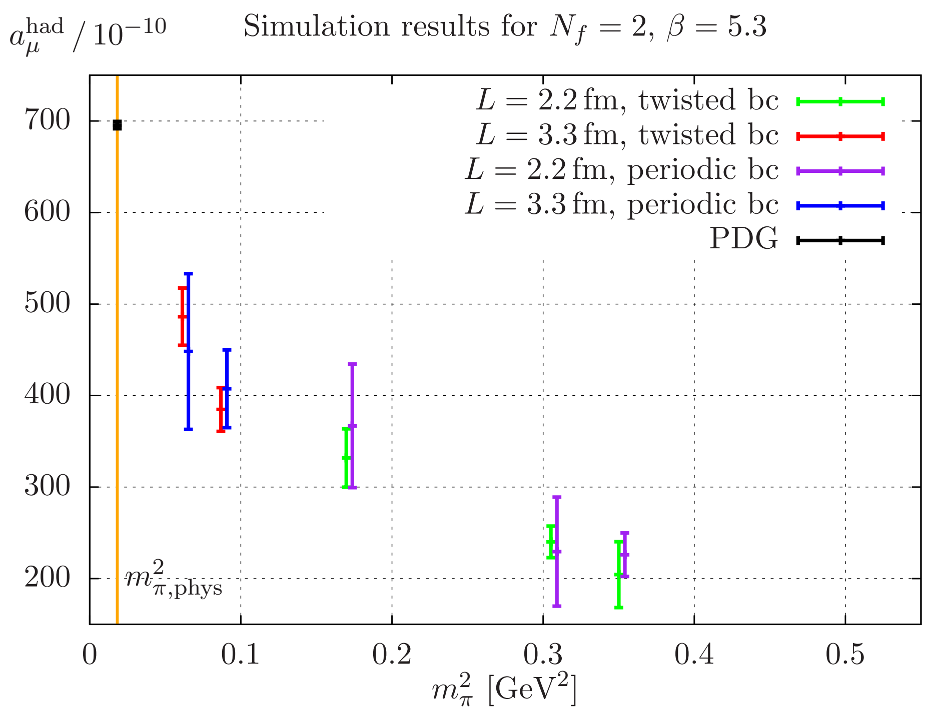




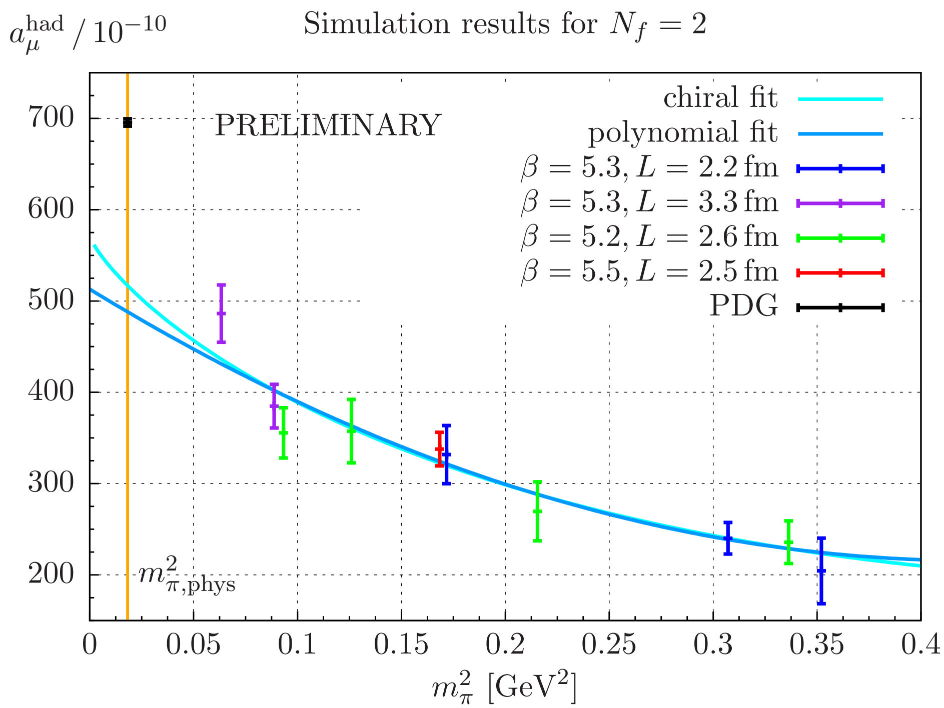
- Shape of convolution function precisely determined using twisted boundary conditions

  $\rightarrow$ statistical errors reduced by factor 2



- Further improvement: higher statistical precision at low q^2

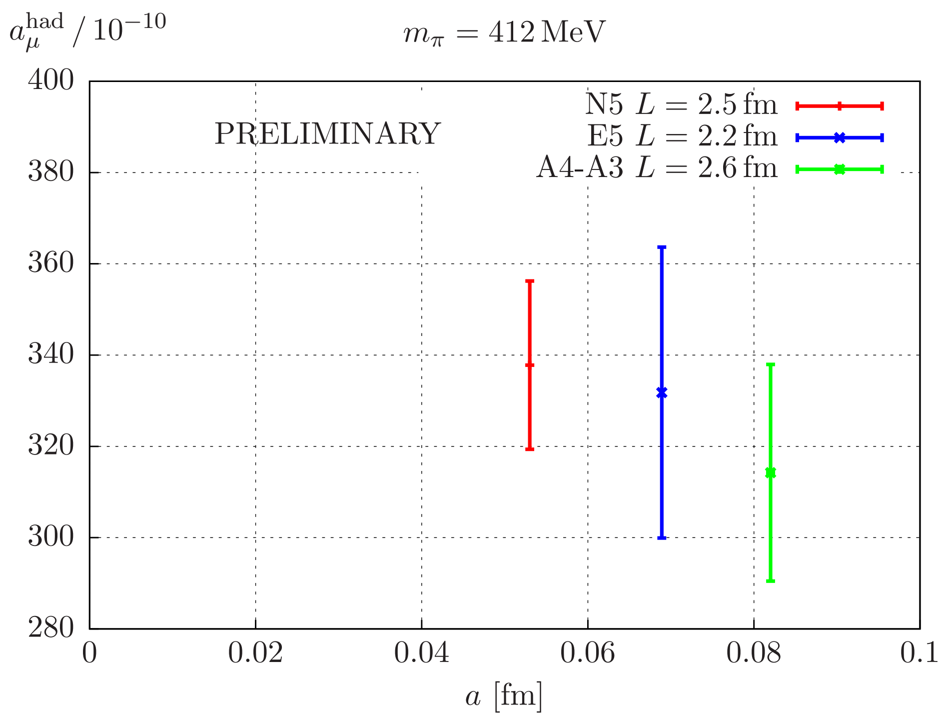
- Strong pion mass dependence of a_μ^{had} :



- Chiral fit: $a_\mu^{\text{had}}(m_\pi^2) = A + Bm_\pi^2 + Cm_\pi^2 \ln m_\pi^2$

- Expect cutoff effects of $\mathrm{O}(a)$, since point-split vector current receives off-shell contributions

- Discretisation errors in a_μ^{had} $(m_\pi = 412\,\mathrm{MeV})$

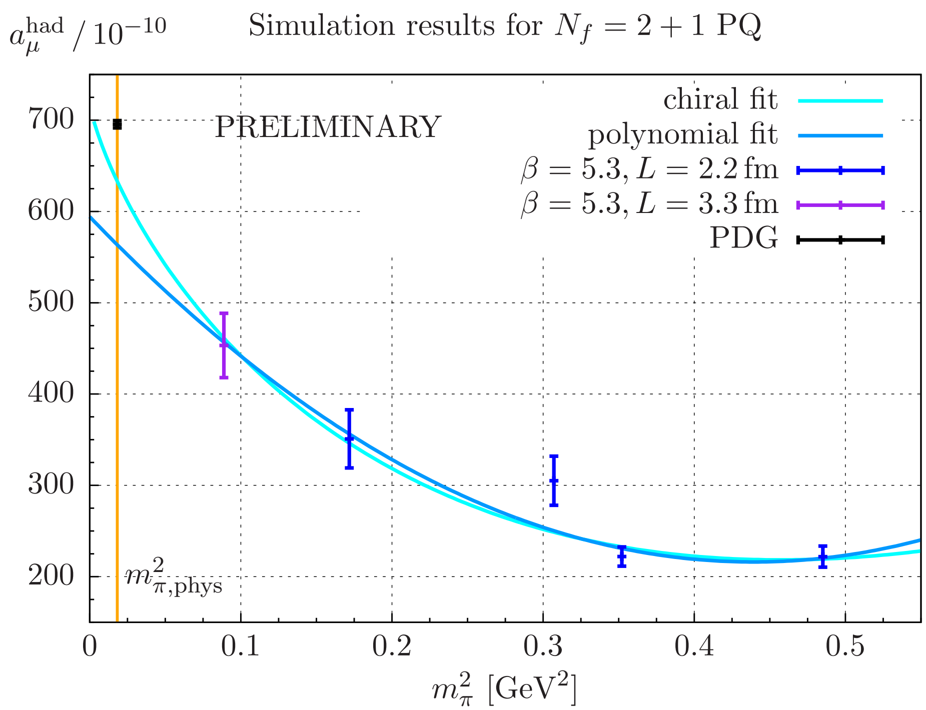


- Lattice artefacts small relative to statistical errors

Adding contribution from quenched strange quark

- Increase by $\approx 20\%$ w.r.t. two-flavour theory

  (single lattice spacing: $\beta = 5.3$)



- Chiral fit: $a_\mu^{\rm had}(m_\pi^2) = A + Bm_\pi^2 + Cm_\pi^2 \ln m_\pi^2$

- More chiral points to be added

4. Summary

- Hadronic vacuum polarisation on the lattice is tough
- This work: several important technical & conceptual improvements:
  - Separate studies of connected & disconnected contributions is justified
  - Twisted boundary conditions valuable tool to reduce statistical and systematic uncertainties
  - Develop & improve methods to compute disconnected contributions

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- Future progress: smaller pion masses, larger volumes
- Current status: not yet in a position to challenge phenomenological approach
- Hadronic light-by-light scattering:

  even tougher problem; new ideas & concepts required