Heavy Quark Potential

from the thermal Wilson Loop in Lattice QCD

Alexander Rothkopf

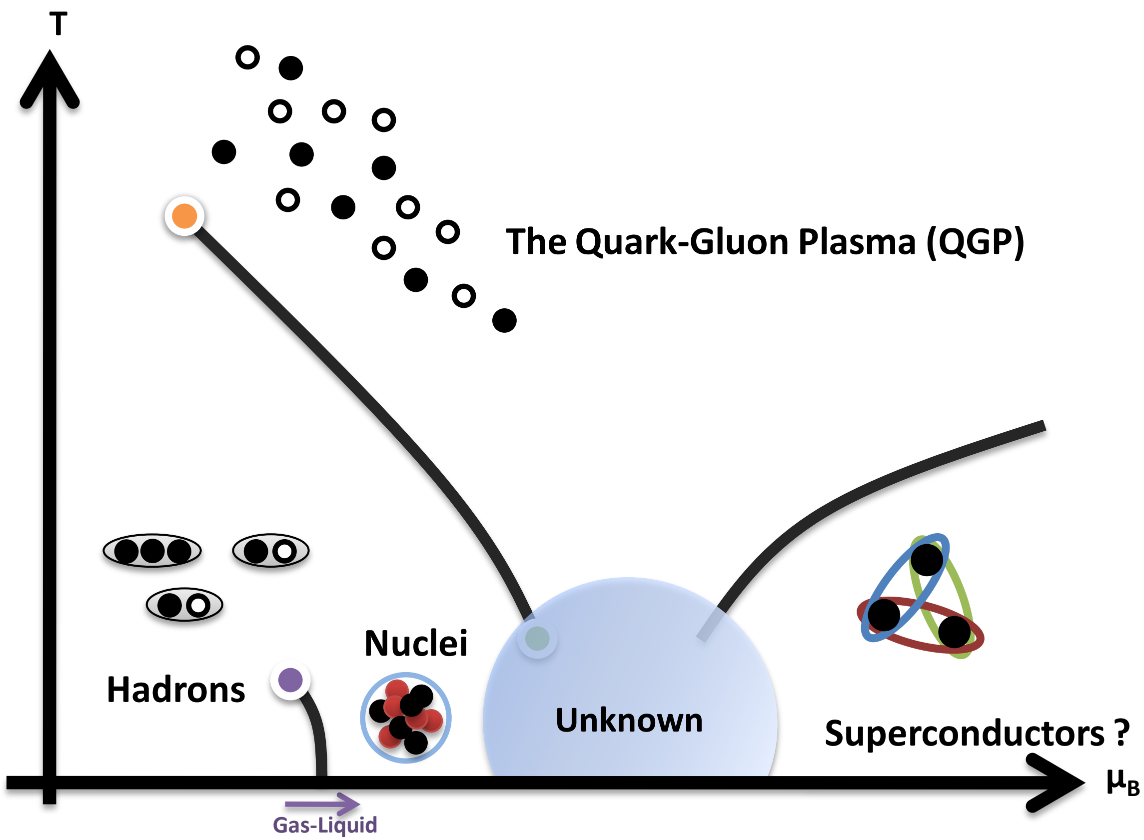
In collaboration with T. Hatsuda & S. Sasaki

Strong interactions: From methods to structures

474th International Wilhelm und Else Heraeus Seminar

東京大学
THE UNIVERSITY OF TOKYO

Phases of QCD



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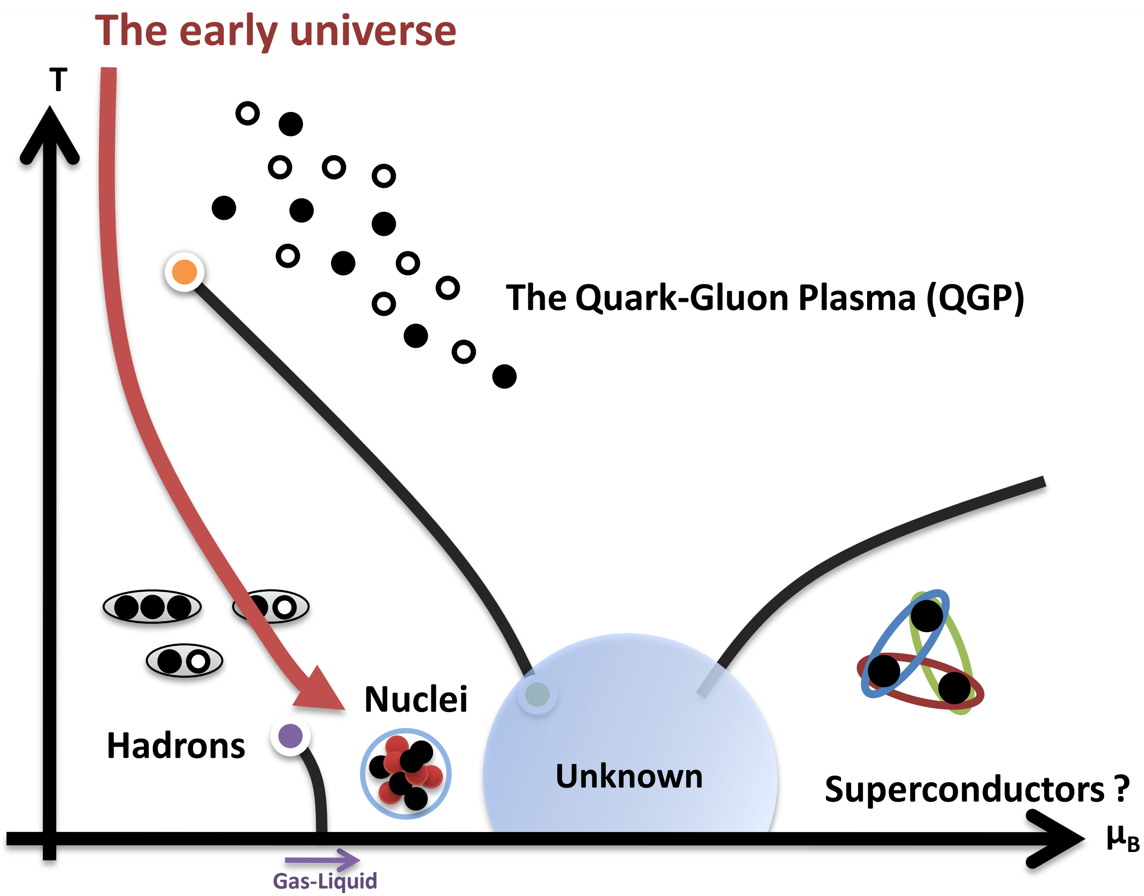

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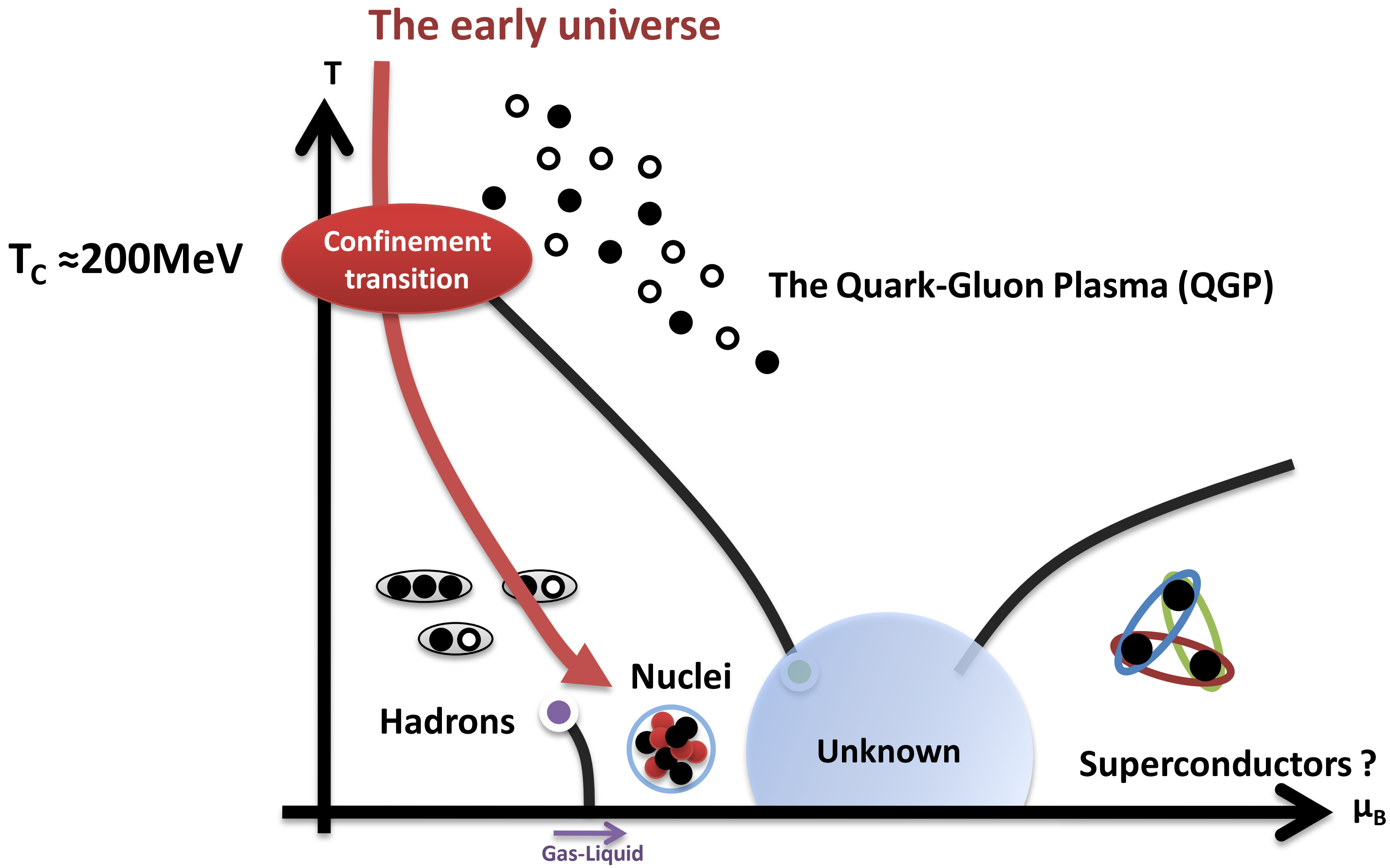

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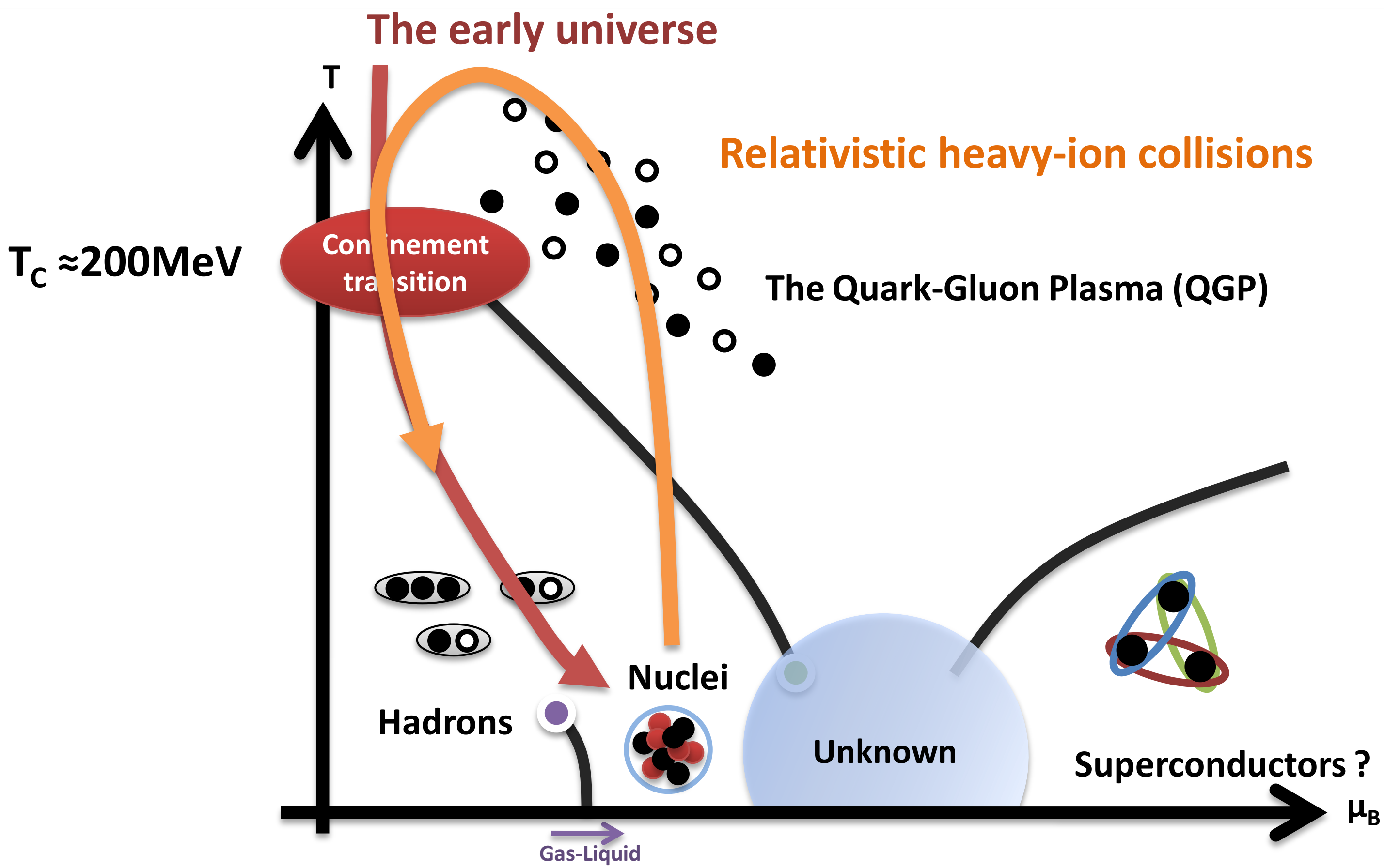

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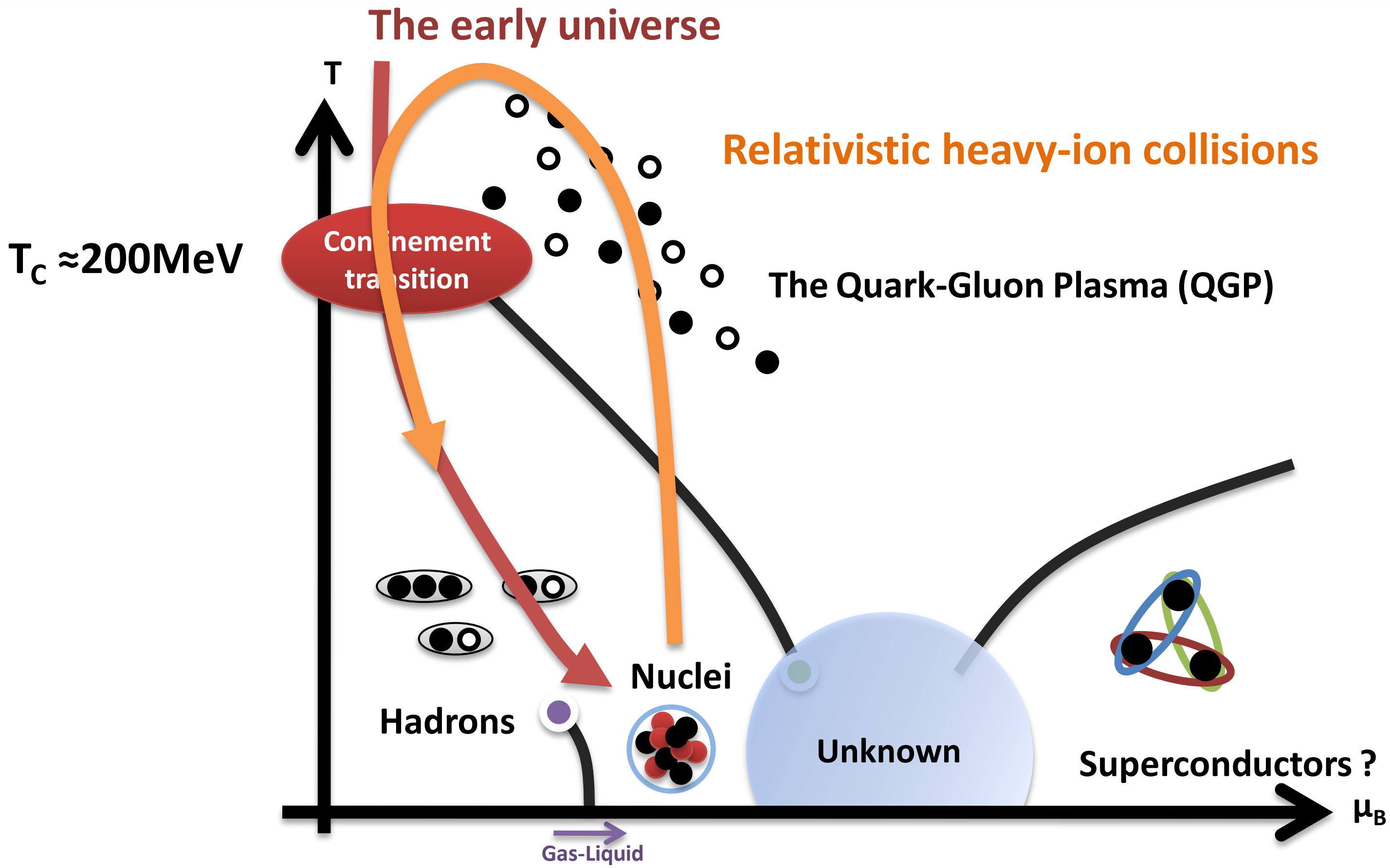

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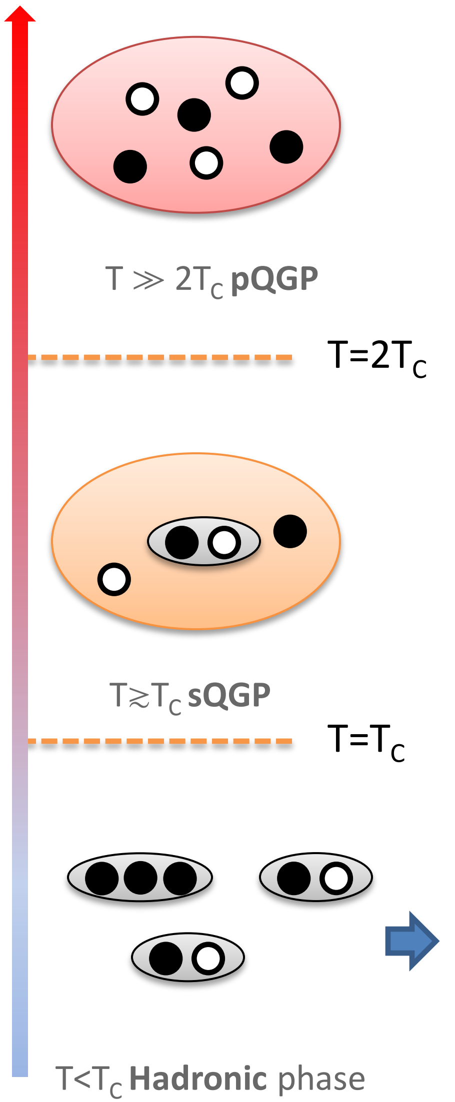
- **Charmonium**: Hadronic thermometer predicted to melt at $1.2T_C$ Matsui, Satz 1986

Current status



T=0 NRQCD & pNRQCD — Reviewed in Brambilla et al. 2005

- Expansion in $1/m_Q$ (Minkowski time):

$$V_{m=\infty}(R) = \lim_{t\to\infty} \frac{i}{t} \log \left\langle \mathrm{Tr}\left(\mathcal{P}_\Box \exp\left[\frac{ig}{c} \int_\Box dx_\mu A^\mu(x) \right] \right) \right\rangle$$

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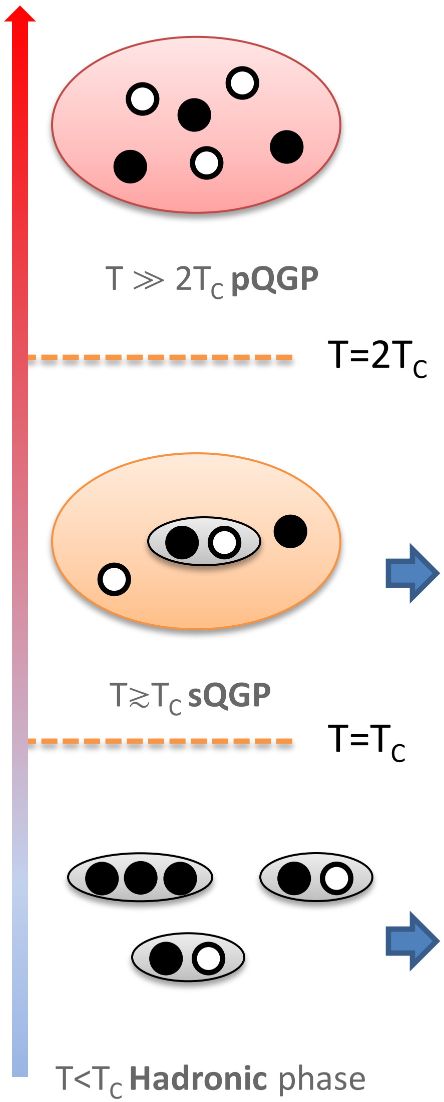

Potential Models

Nadkarni, 1986

- **Ad-hoc** choice: Free Energies or Internal energies
- **No Schrödinger equation**
- Questions: Gauge dependence Entropy contributions

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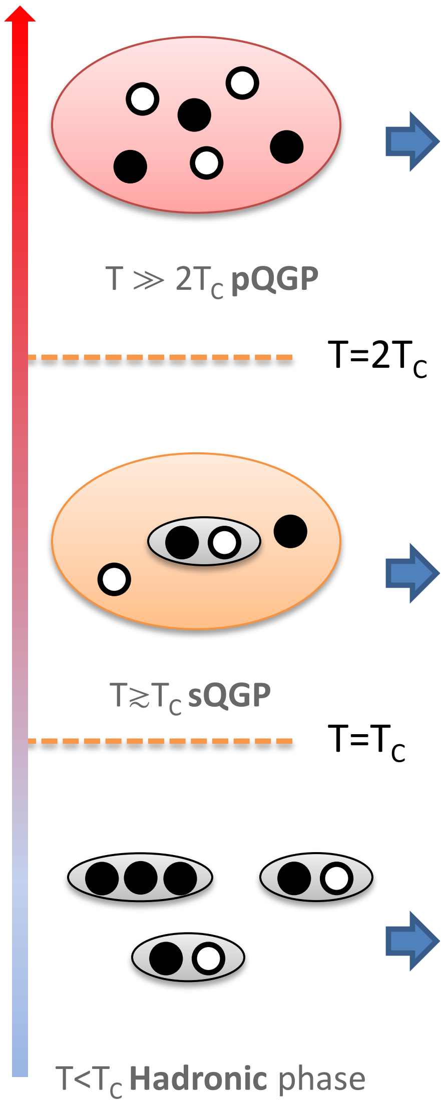

Hard Thermal Loop
Laine et. al. 2007

- Real-time formulation based on the forward correlator
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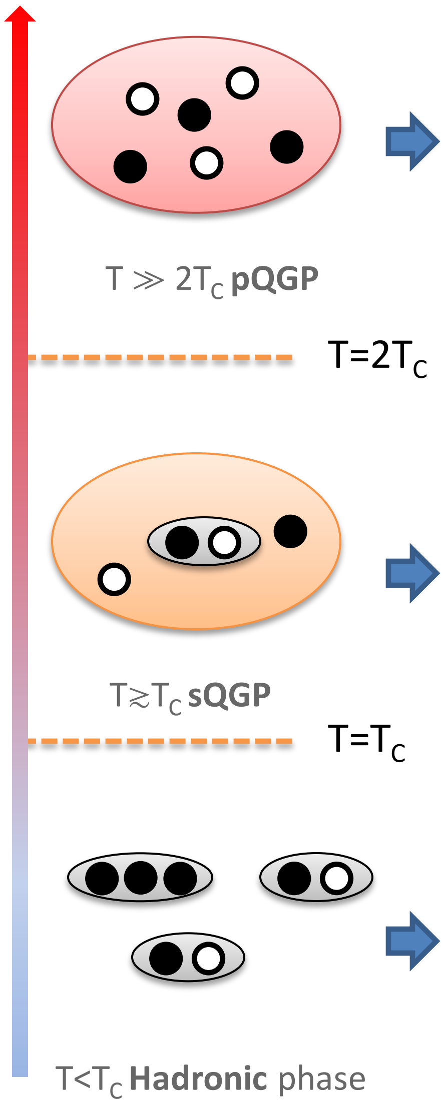

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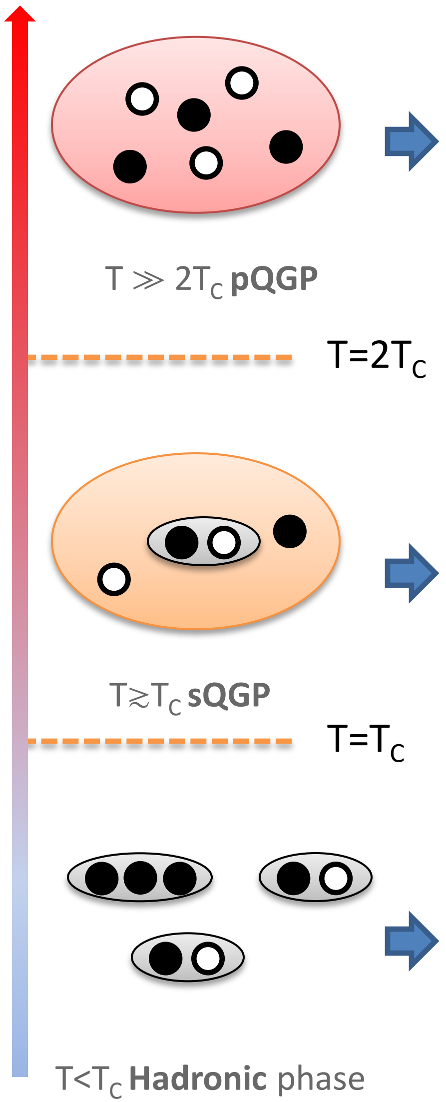

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Spectra from LQCD

Asakawa, Hatsuda, Nakahara 2001

- Obtained from non-perturbative Lattice QCD using Bayesian inference
- **Maximum Entropy Method:** numerically difficult but well established
- Identification of bound state by eye only

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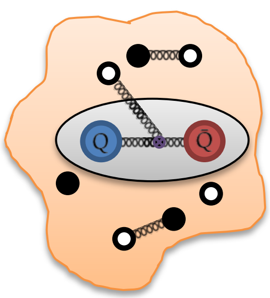
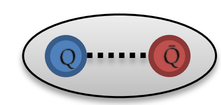
Derivation of the heavy quark potential →

Quantum mechanics





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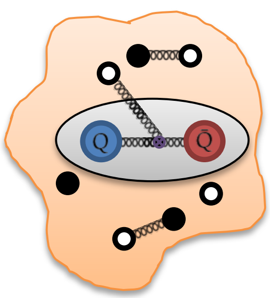

Dirac fields

$\bar{Q}(x), Q(x)$

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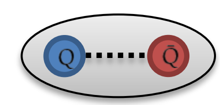

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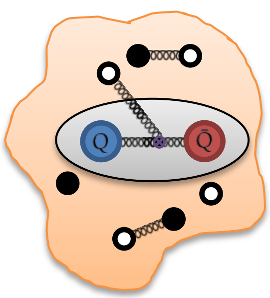

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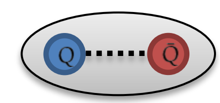

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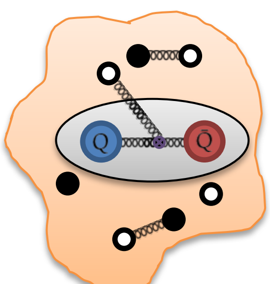

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$V^{(0)}(R) \leftarrow \rho_\Box(\omega, R)$

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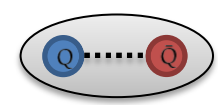

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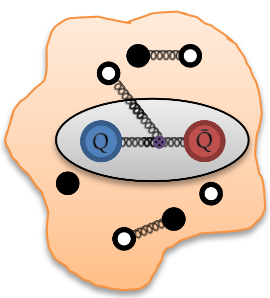

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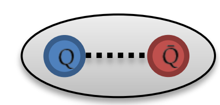

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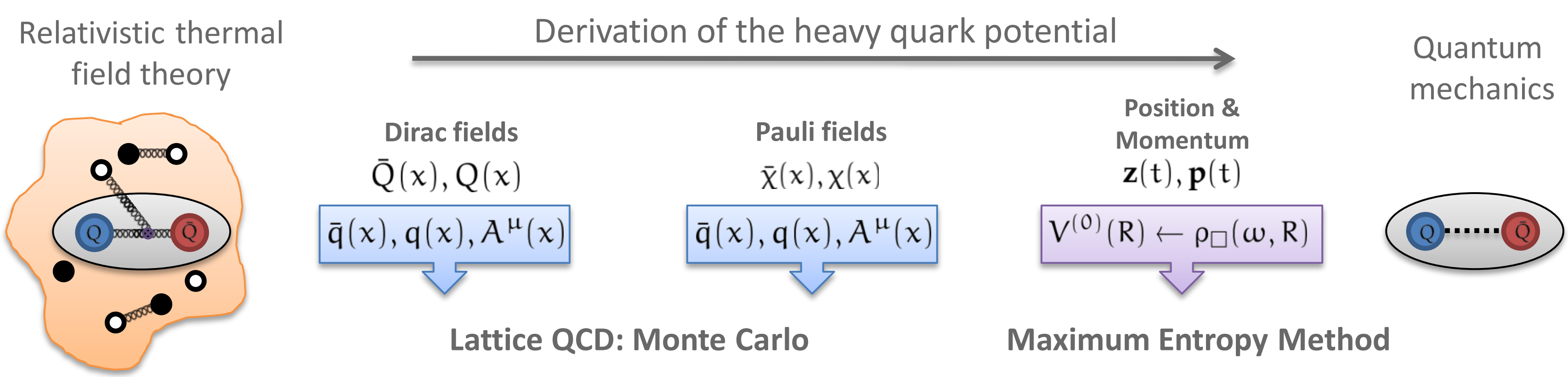

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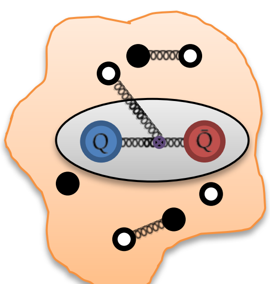

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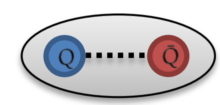

Lattice QCD: Monte Carlo | **Maximum Entropy Method**

  - Derive a Schrödinger equation with a **non-perturbative**, spin-independent potential

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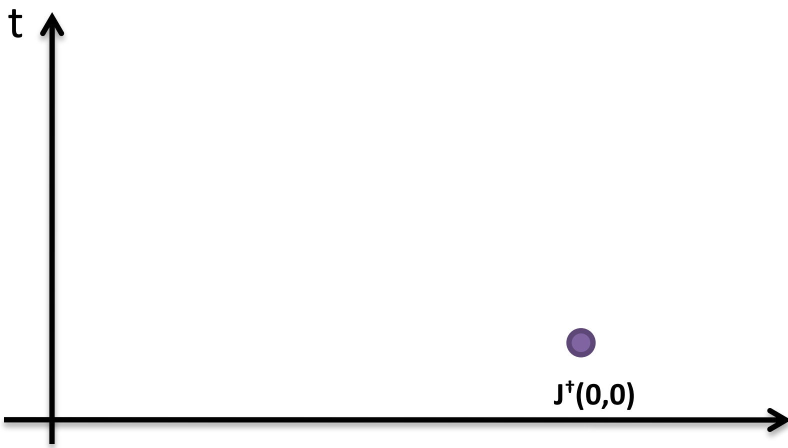

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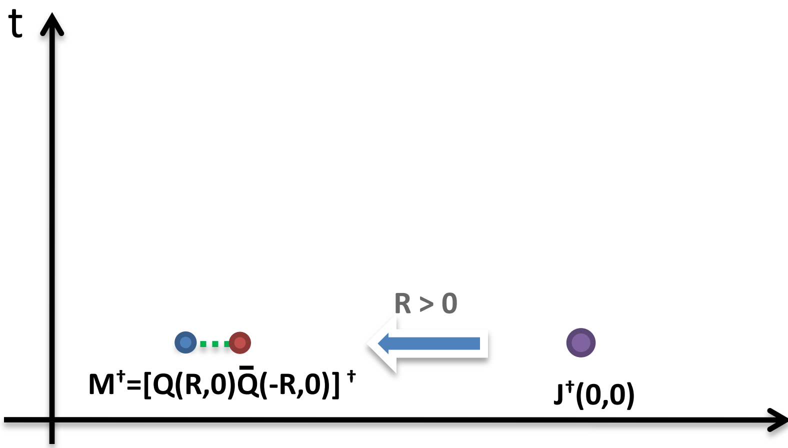
$$M = \bar{Q}(x)\Gamma W(\mathbf{x},\mathbf{y})Q(y)$$



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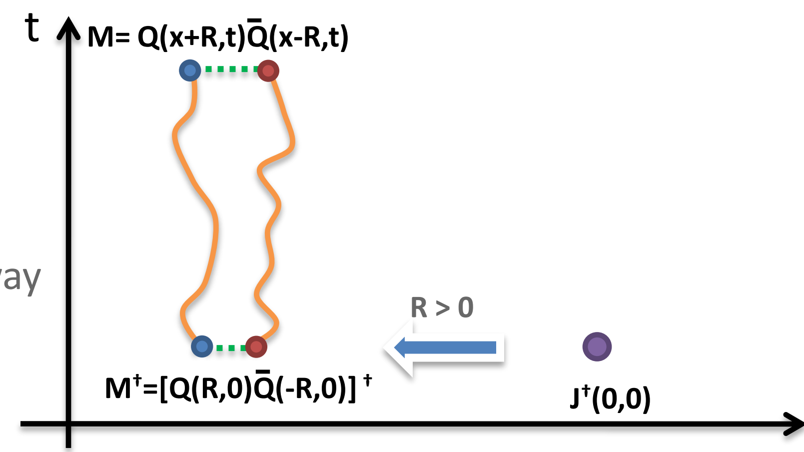

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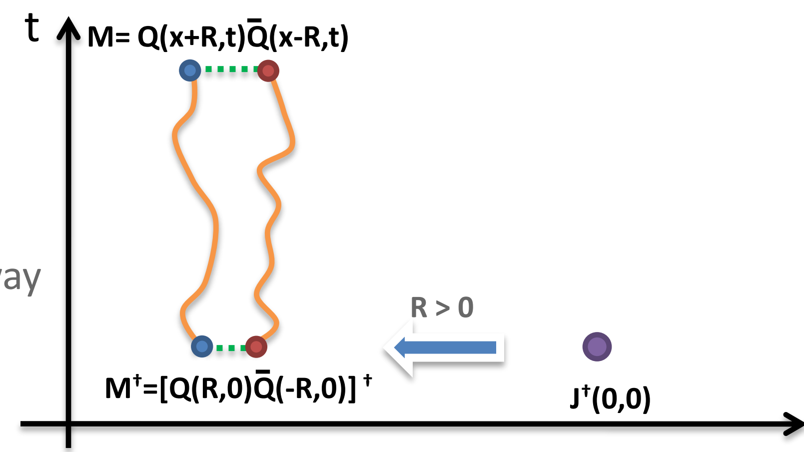

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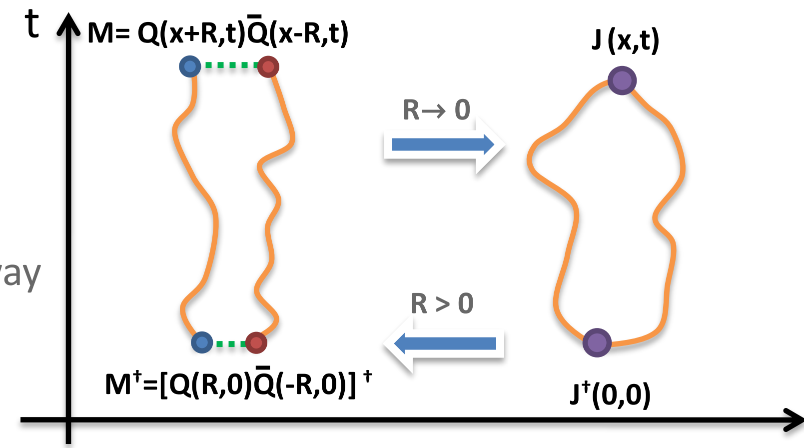

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  - Only naturally gauge invariant quantity: current - current correlator

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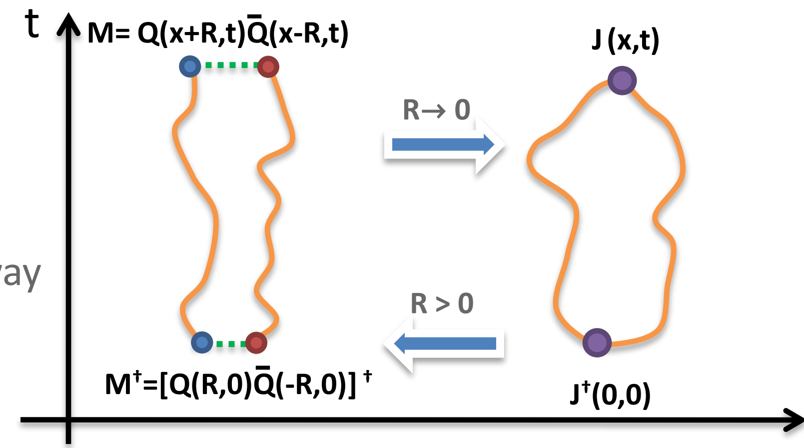

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  - Use separation of scales to simplify the expression for $D^{>}(\mathbf{R},t)$

$$D^{>}(\mathbf{R},t) = \left\langle \mathcal{T}\left[\int \mathcal{D}[\bar{Q},Q]\ \Gamma\bar{\Gamma}\ WW^{\dagger}\ Q(y')\bar{Q}(y)Q(x)\bar{Q}(x')\ e^{iS_{QQ}[Q,\bar{Q},A]}\right]\right\rangle_{q,\bar{q},A}$$

Integrating out $E=m_Qc^2$

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- Replace the degrees of freedom for the heavy fermions Q -> Q = (χ,ξ)

$$S_{QQ}[A] = \bar{Q}(x)\Big(i\gamma^\mu D_\mu(x;A) - mc\Big)Q(x)$$

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  - FTW Transformation leaves the path integral unchanged at this order

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⇓

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2x2 sub matrix

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q.m. heavy quark **Green`s functions** (2x2)

Q.M. Path Integrals

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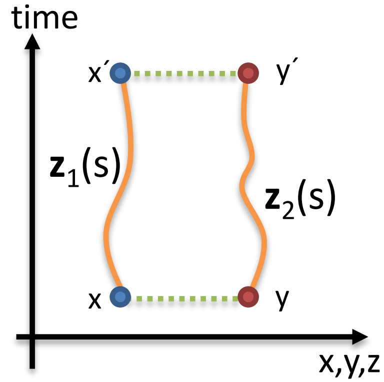

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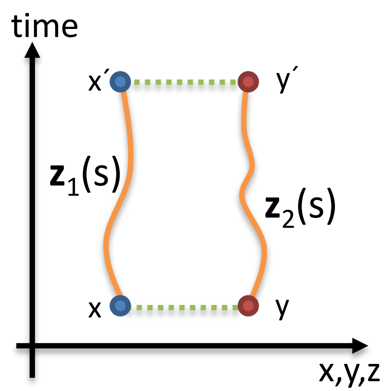
This is not just the rectangular Wilson loop: fluctuating paths



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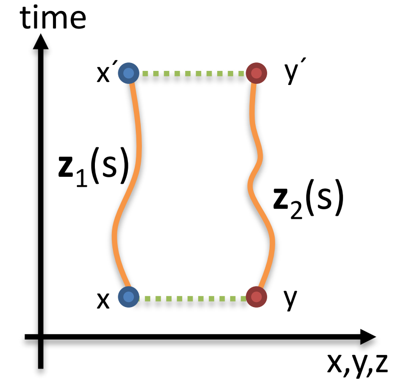

- To read off the **Hamiltonian for the two-body system** we need to rewrite:

$$\langle \mathrm{Tr}[\exp\left[\oint A\right]]\rangle \equiv \exp\left[i\int_t^{t'} ds\; U(\mathbf{z}_1(s),\mathbf{z}_2(s),\mathbf{p}_1(s),\mathbf{p}_2(s),s)\right]$$

The Proper Static Potential

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- Systematic expansion of the potential in p/mc: Barchielli (1988) uses v/c instead

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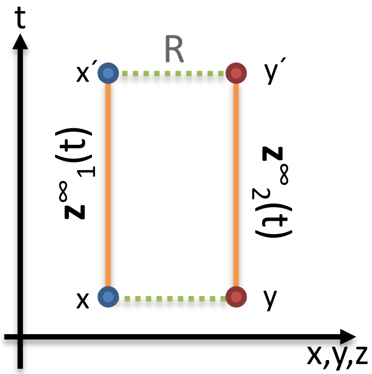

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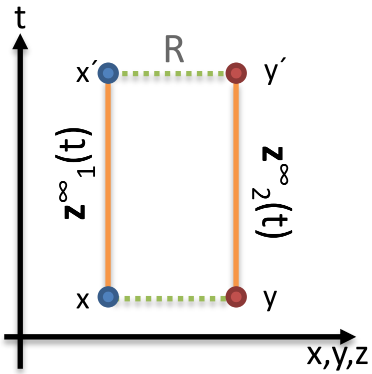

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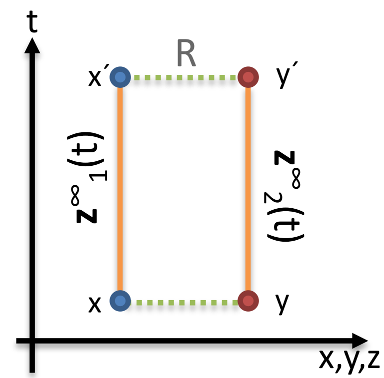

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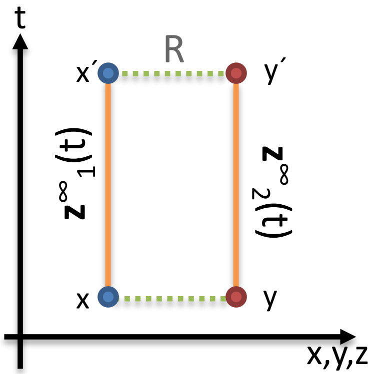

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The Proper Static Potential

東京大学 University of Tokyo

- Systematic expansion of the potential in p/mc: Barchielli (1988) uses v/c instead

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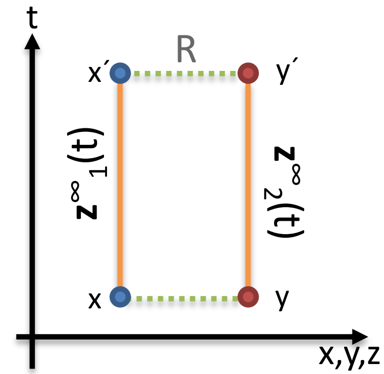

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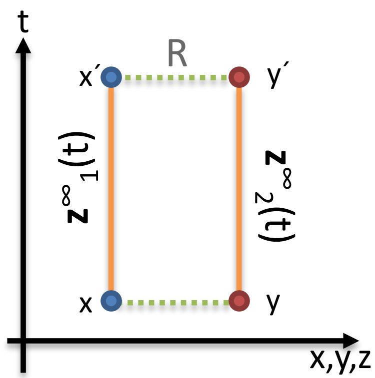

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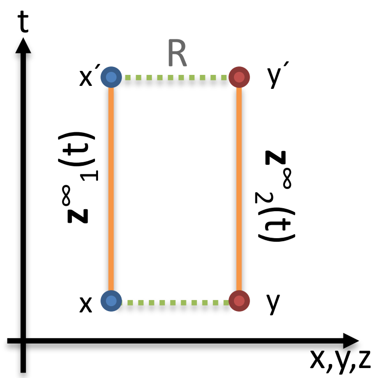

Well defined peaks

$$u(R,t) = \frac{i\partial_t W_\square(R,t)}{W_\square(R,t)} = \frac{1}{W_\square(R,t)}\int d\omega e^{-i\omega t}\,\omega\,\rho_\square(R,\omega) \stackrel{?}{=} \frac{V(R,t)}{W_\square(R,t)}\int d\omega e^{-i\omega t}\rho_\square(R,\omega)$$

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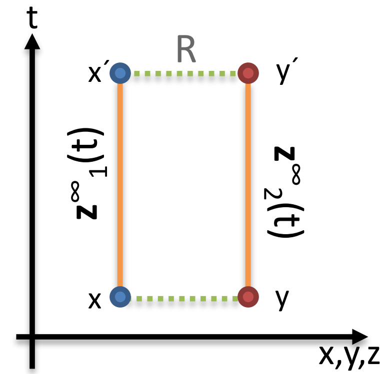

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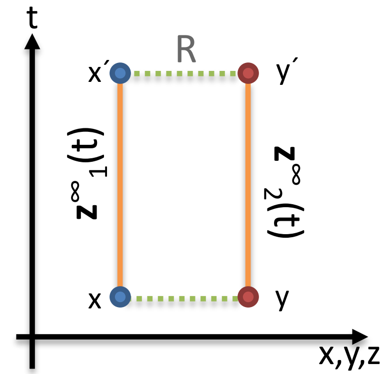

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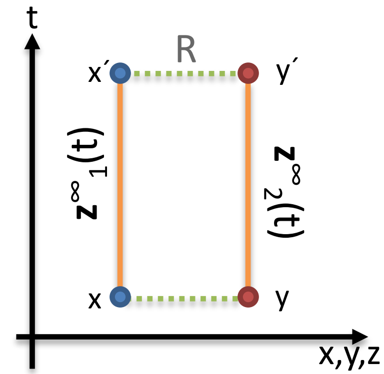

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  - **Intuitive**: Two analytically solvable cases (Breit-Wigner & Gaussian)

The Proper Static Potential

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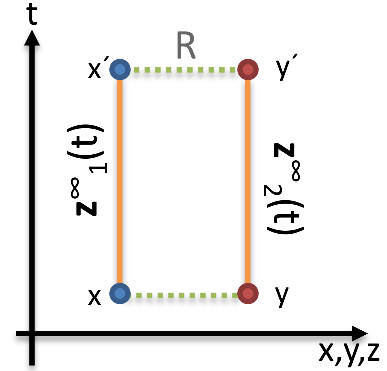

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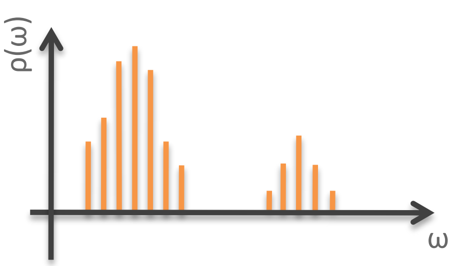

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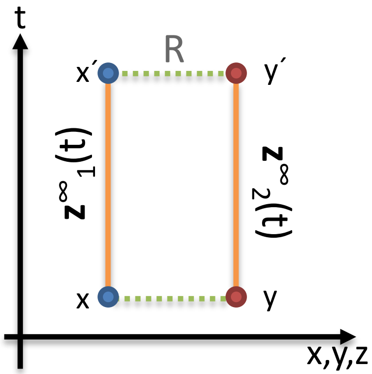

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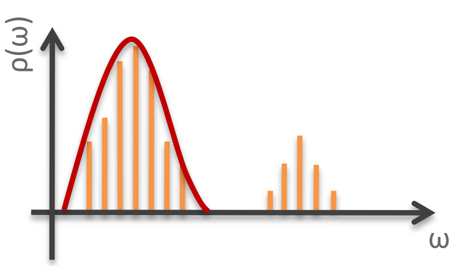

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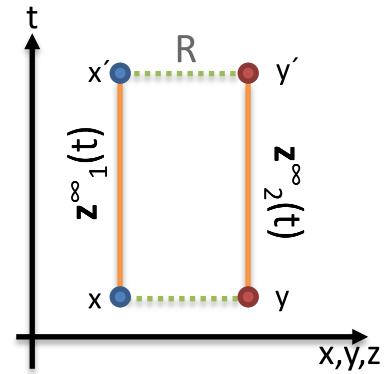

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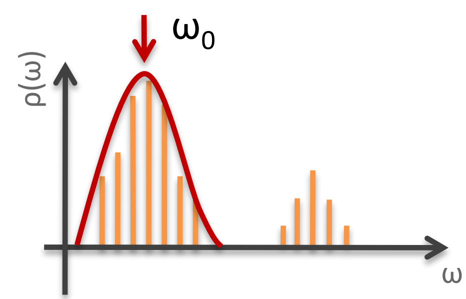

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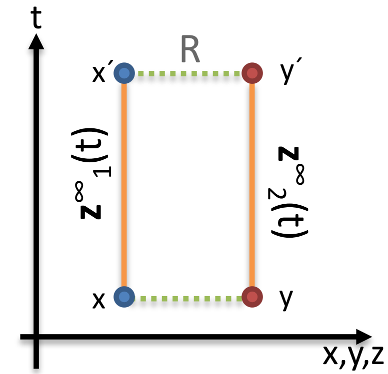

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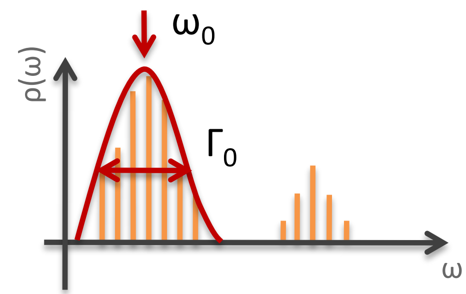

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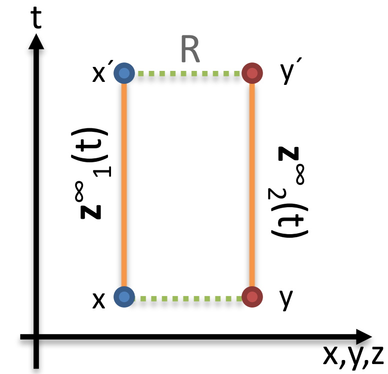

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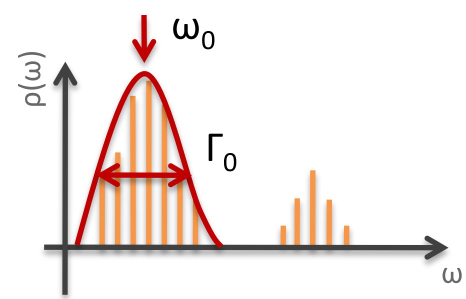

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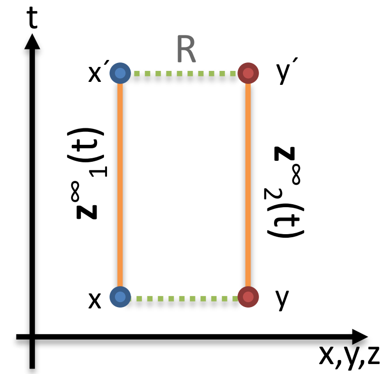

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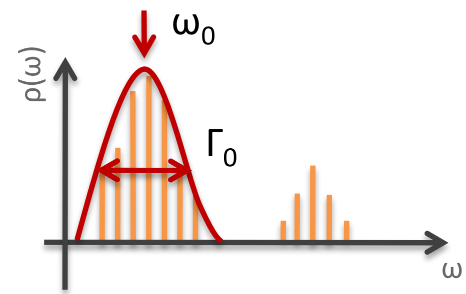

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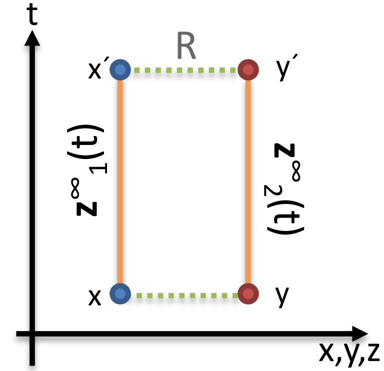

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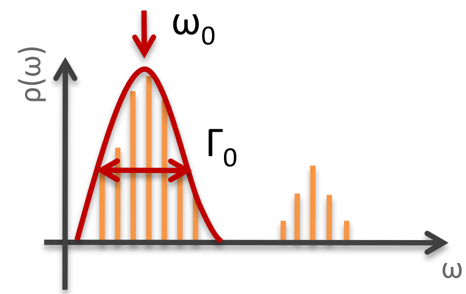

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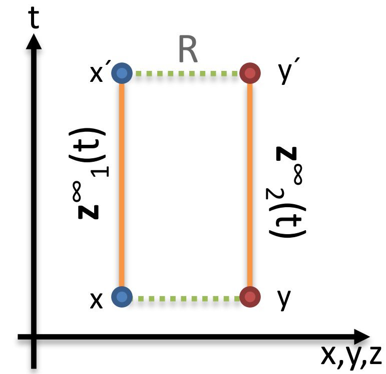

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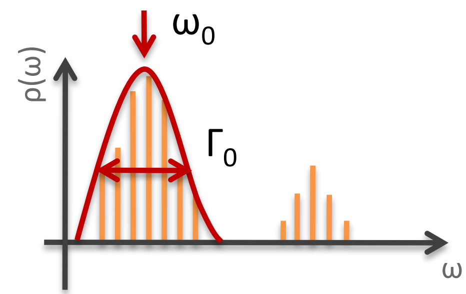

Towards finite momentum

- Still a **static** Schrödinger equation:

$$i\partial_t D^{>}(R,t) = \Big[2mc^2 + \mathrm{Re}V^{(0)}(R,t) + i\mathrm{Im}V^{(0)}(R,t)\Big] D^{>}(t,R) \qquad \text{at} \quad \mathcal{O}\Big(\frac{1}{m^0}\Big)$$

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  - Position z(t) and momentum p(t) are **independent** in Hamiltonian formalism

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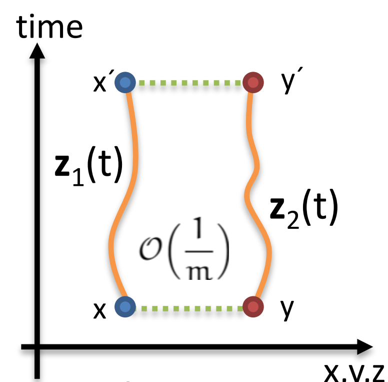

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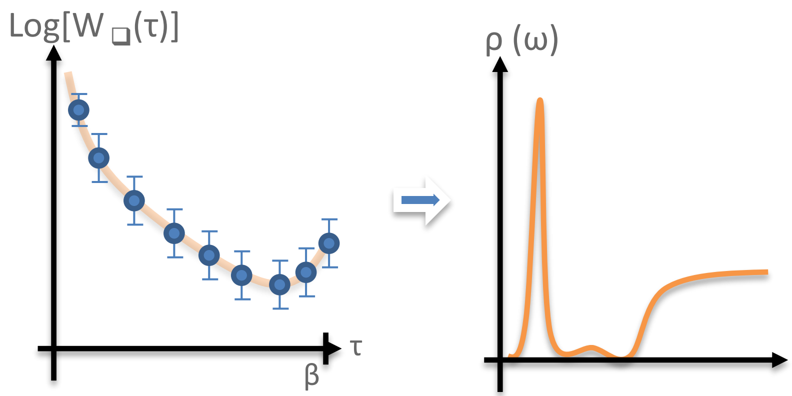

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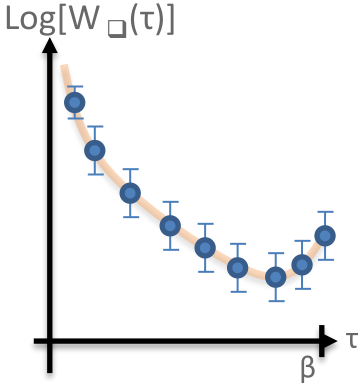

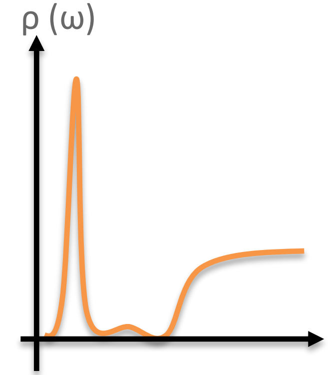

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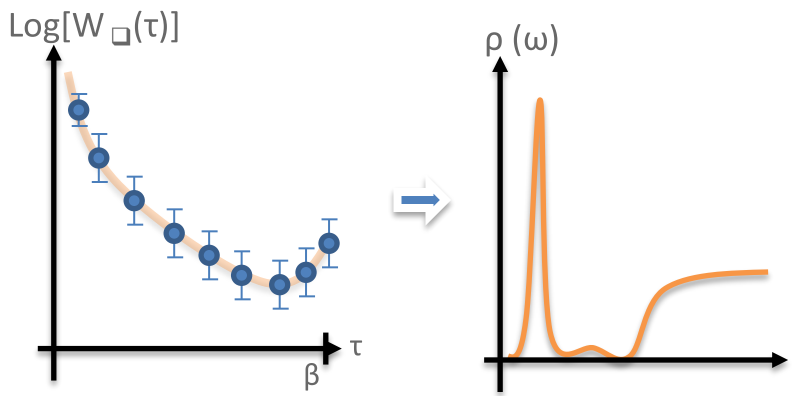

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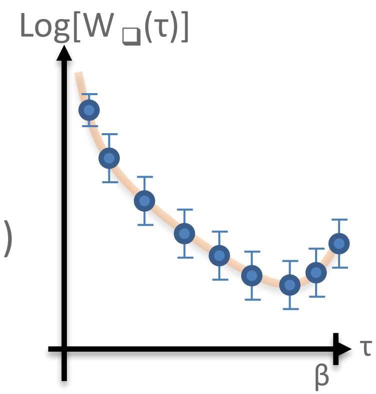

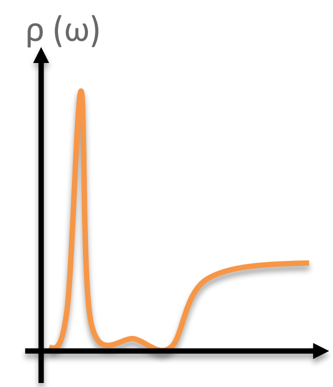

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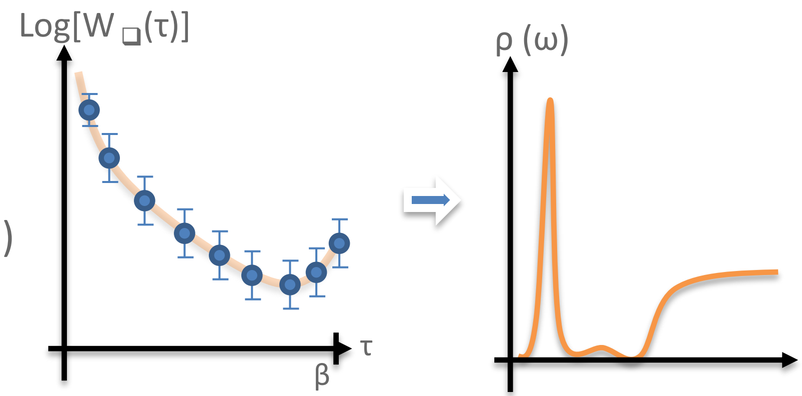

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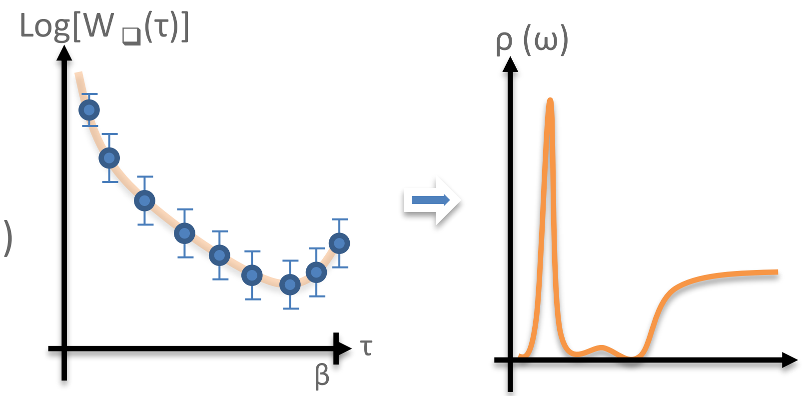

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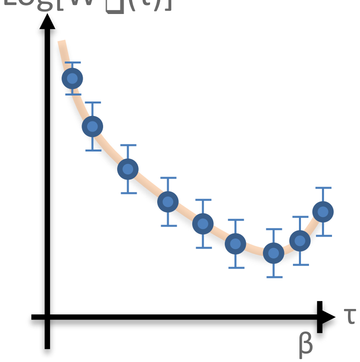

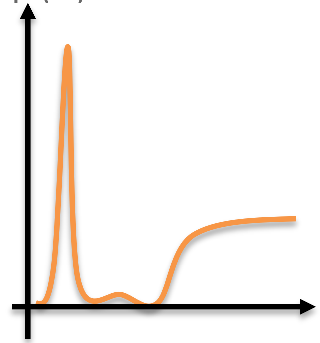

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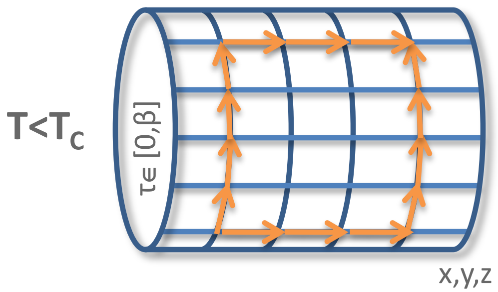

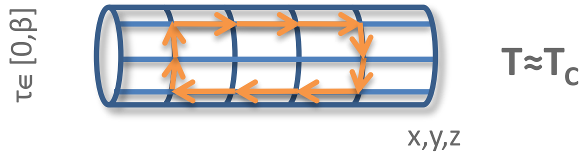

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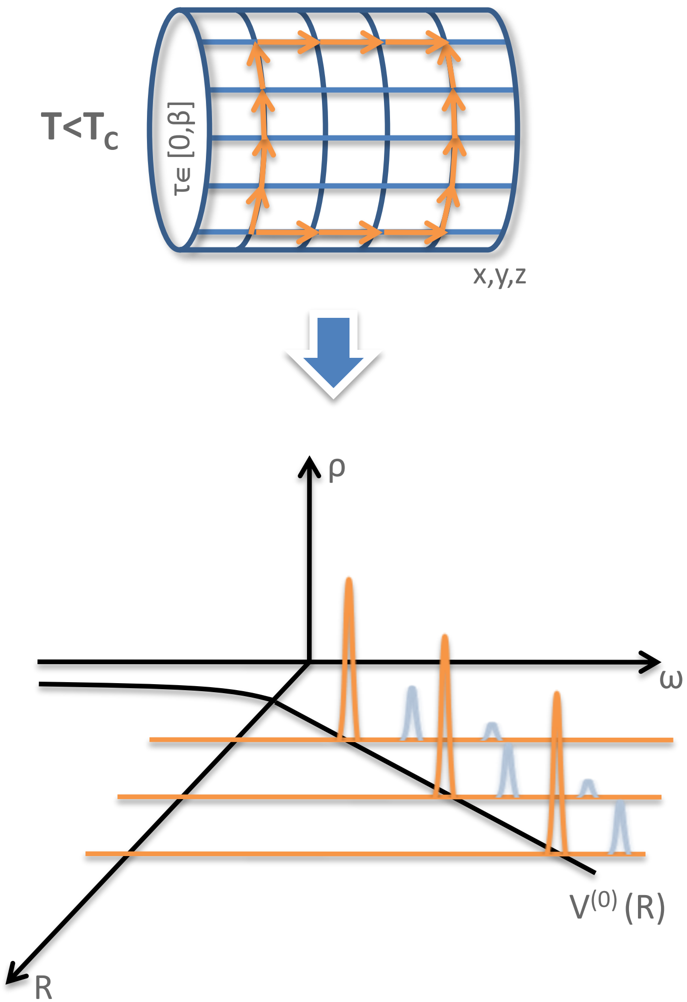

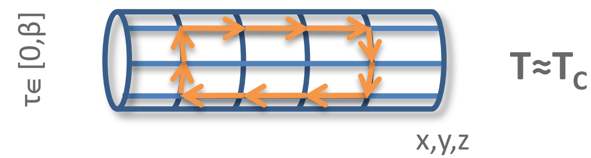

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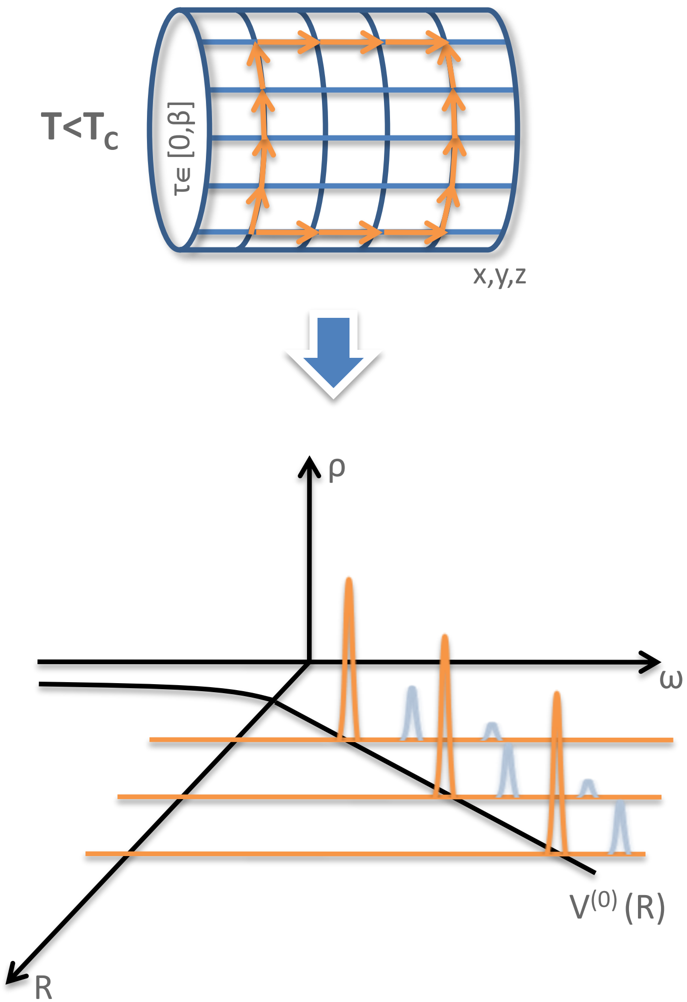

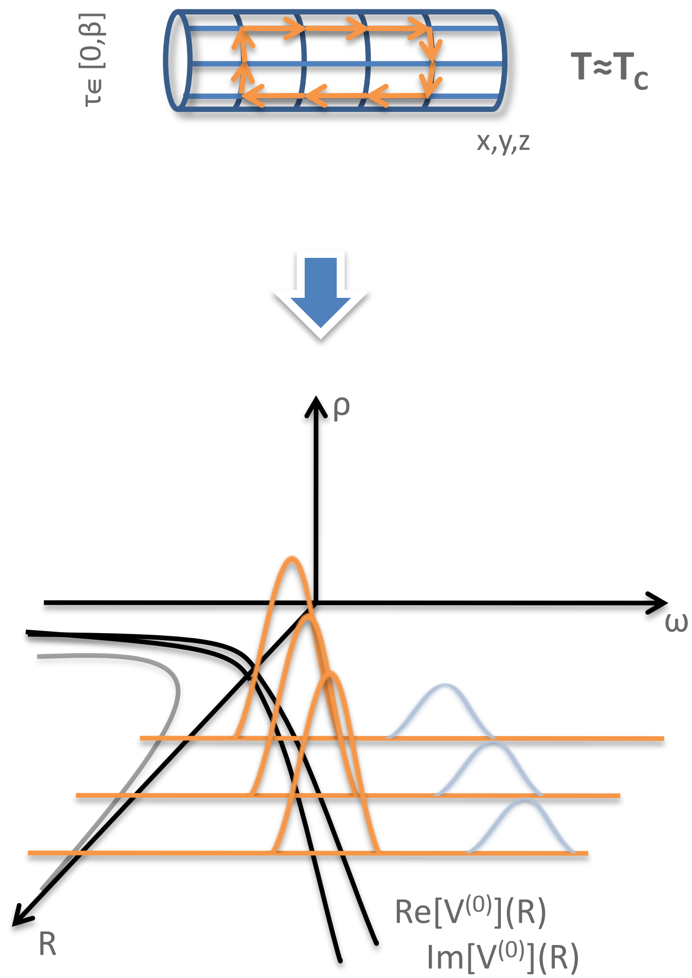

Numerical Results: $T=0.78T_C$

Quenched QCD Simulations

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- NX=20 NT=36 β=6.1 ξ_b=3.2108
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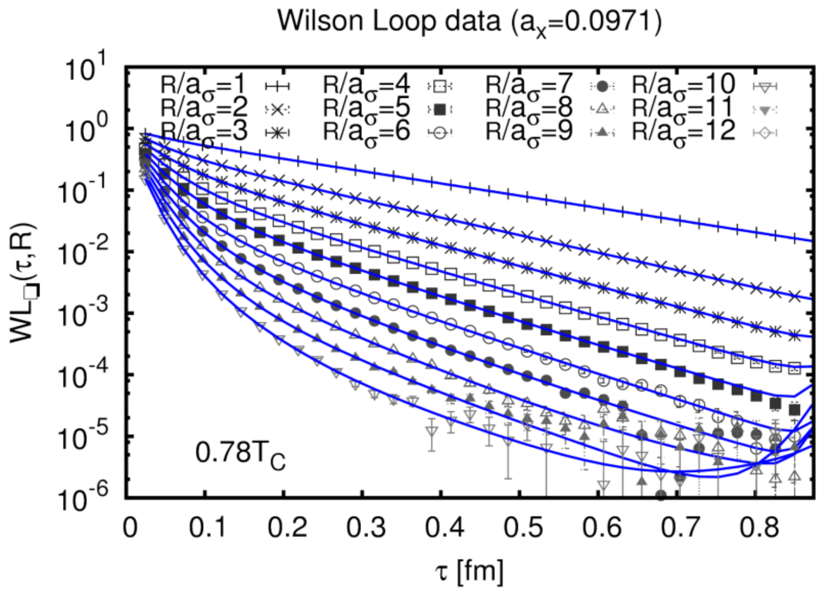

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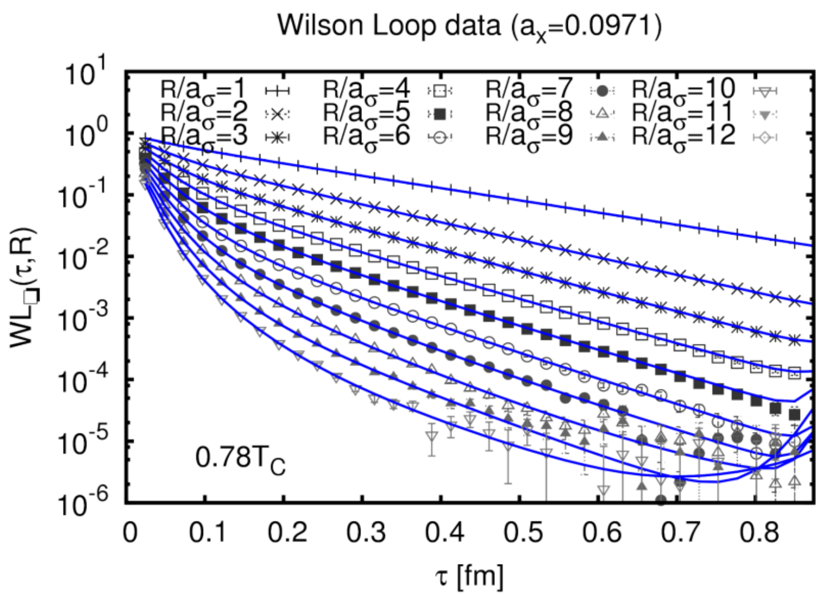

Note that the Wilson Loop is non-symmetric since heavy quarks are not thermalized

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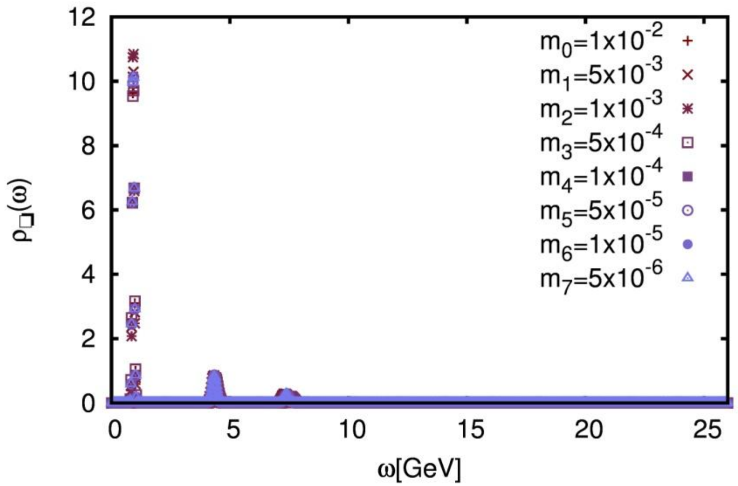
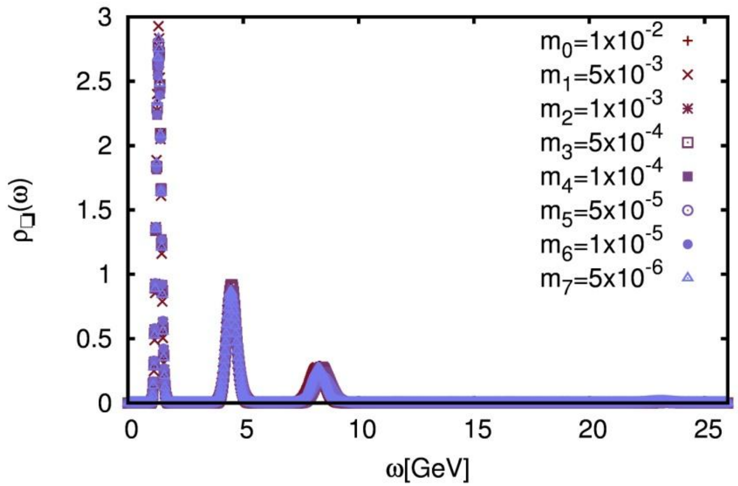


R=0.2fm



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Assess prior dependence

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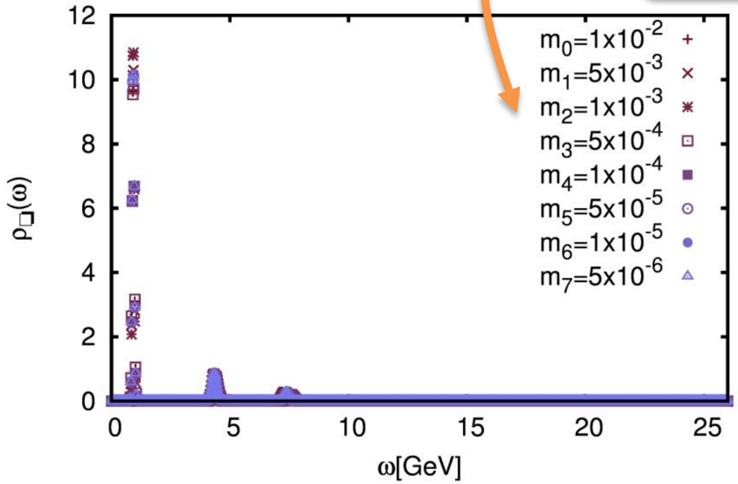

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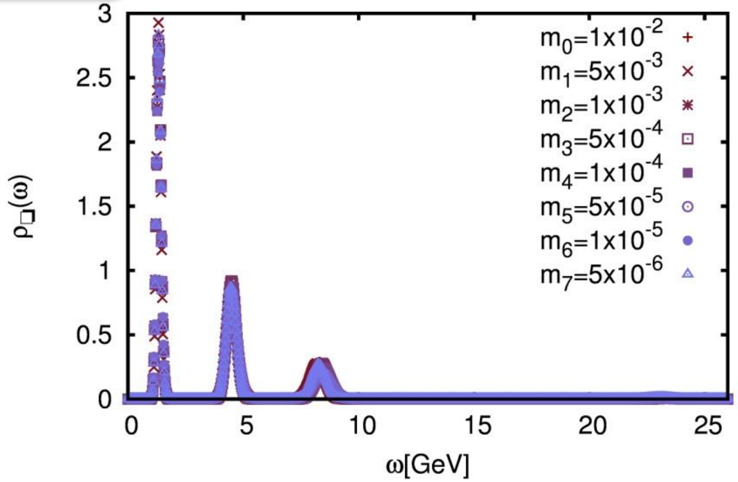

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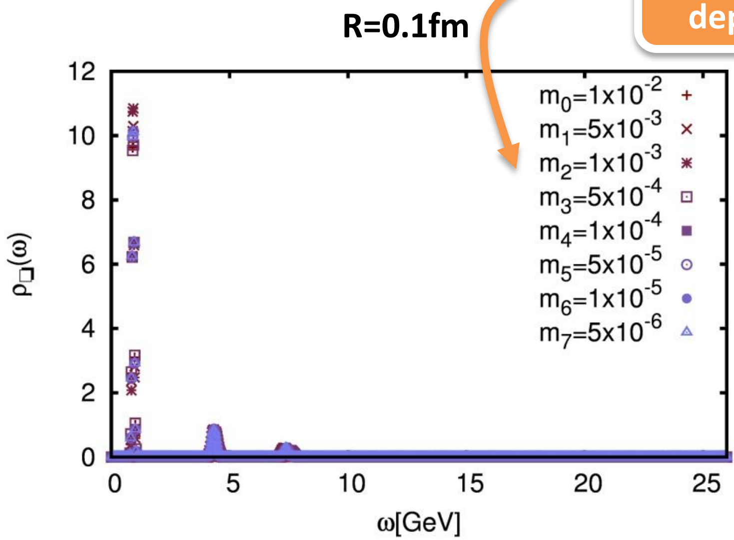

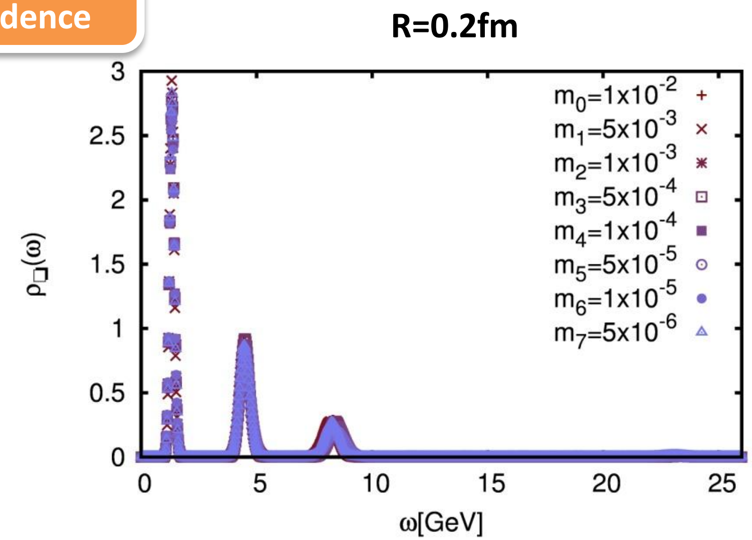

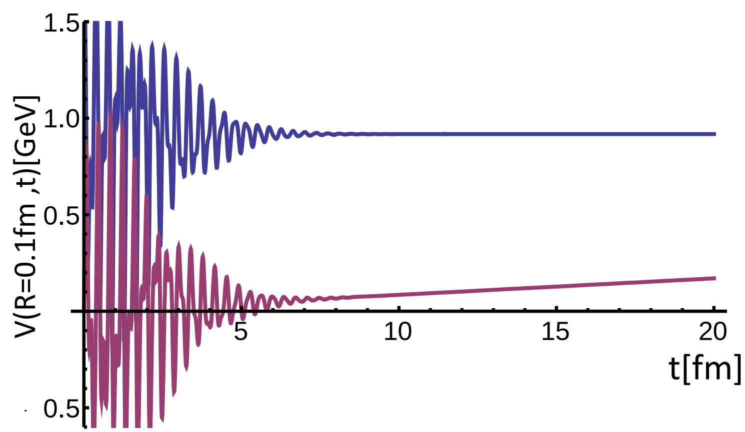

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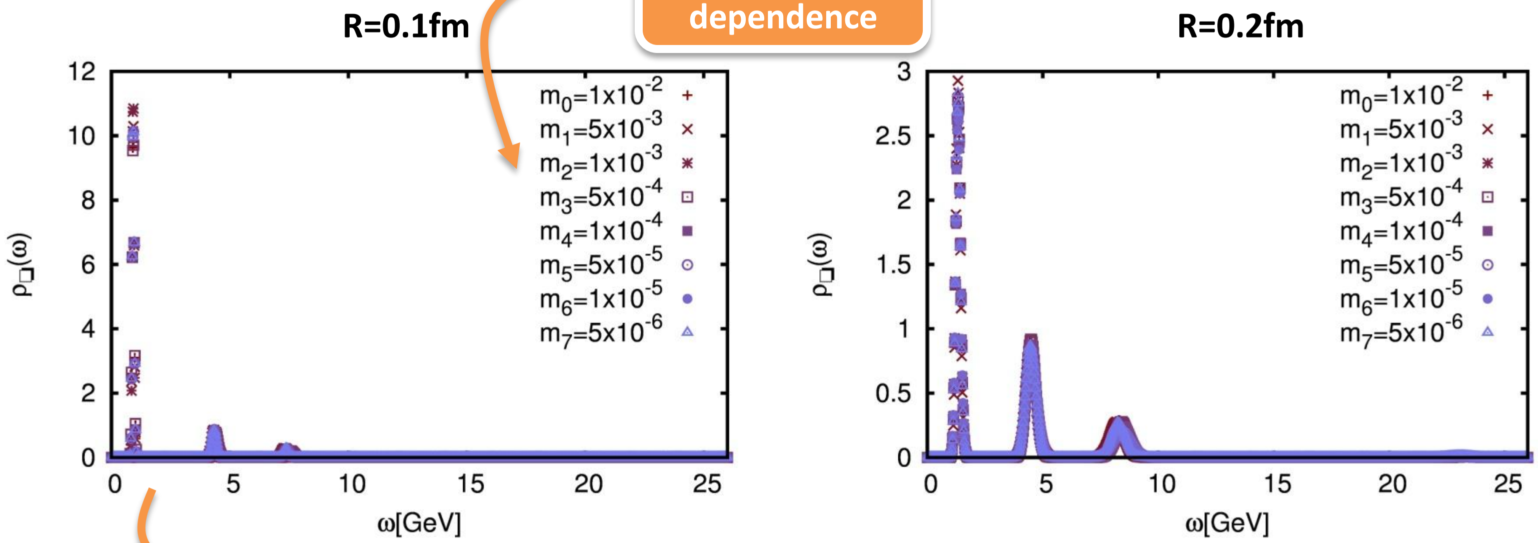
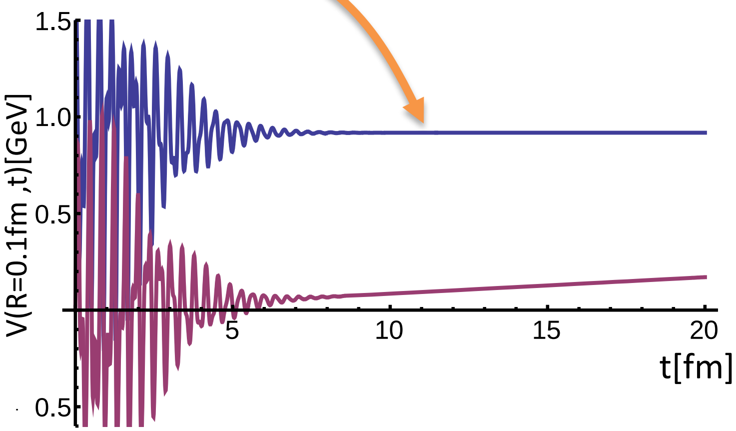


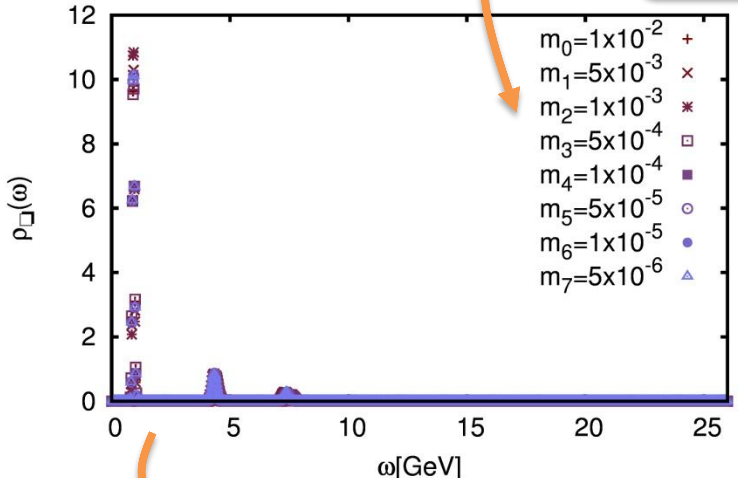
Lowest peak dominance for late times



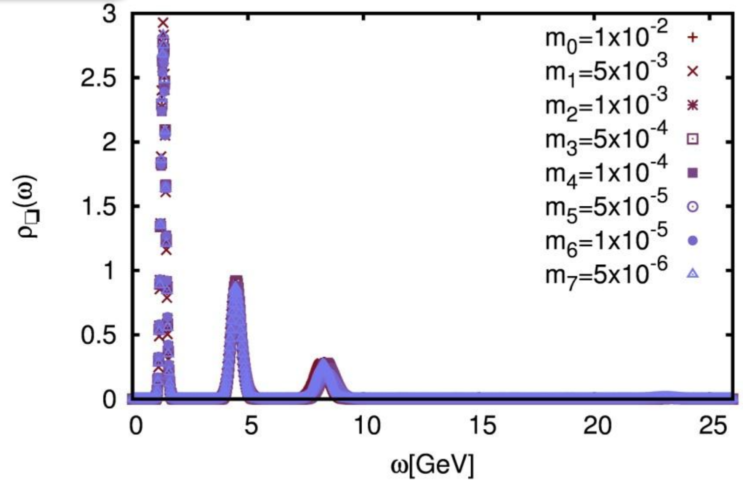
Numerical Results: $T=0.78T_C$

Assess prior dependence

R=0.1fm



R=0.2fm



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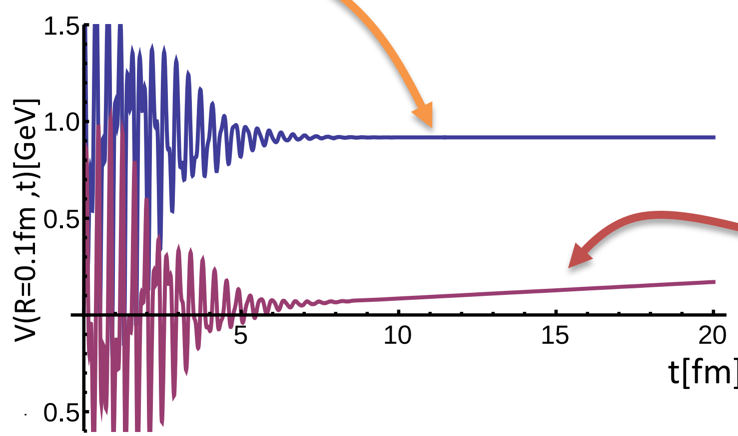

$T<T_C$: Only scattering with glueballs (Im[V] is probably MEM artifact)

Numerical Results: T=0.78T_C

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Assess prior dependence

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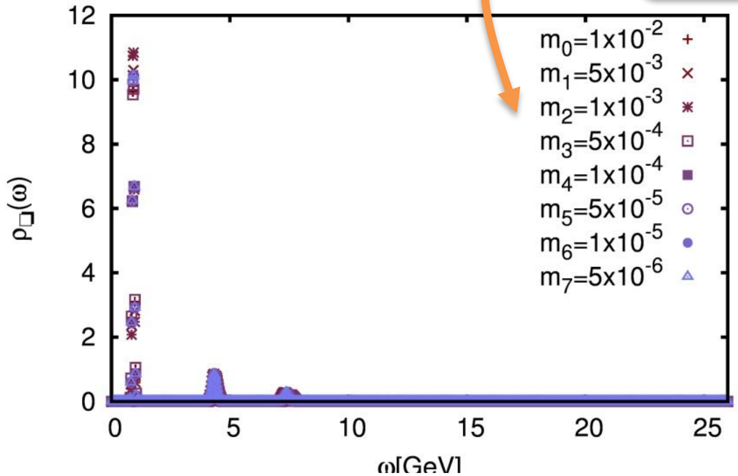

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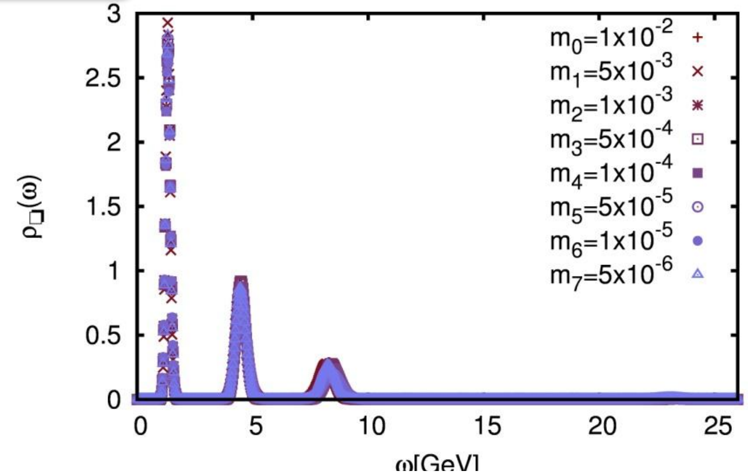
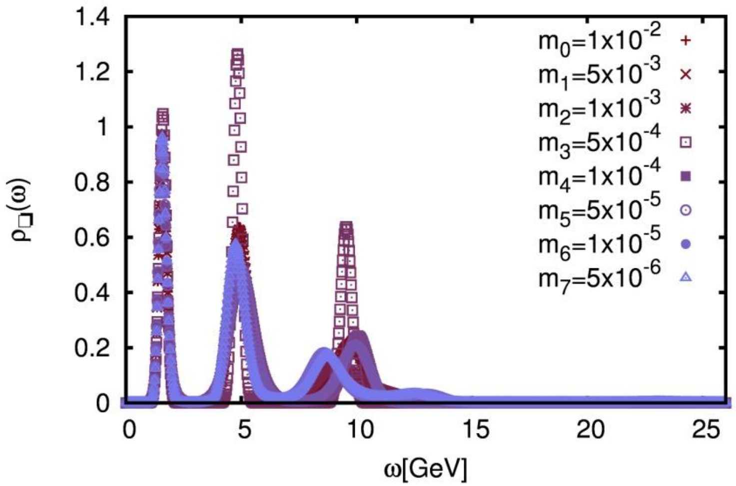
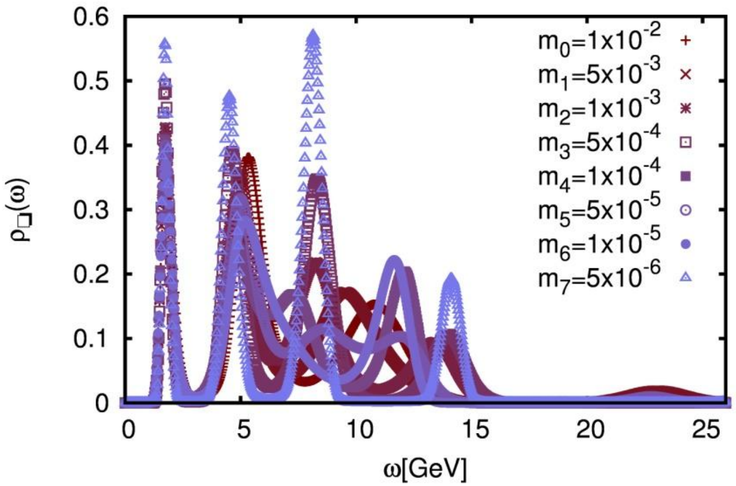


R=0.3fm



R=0.4fm



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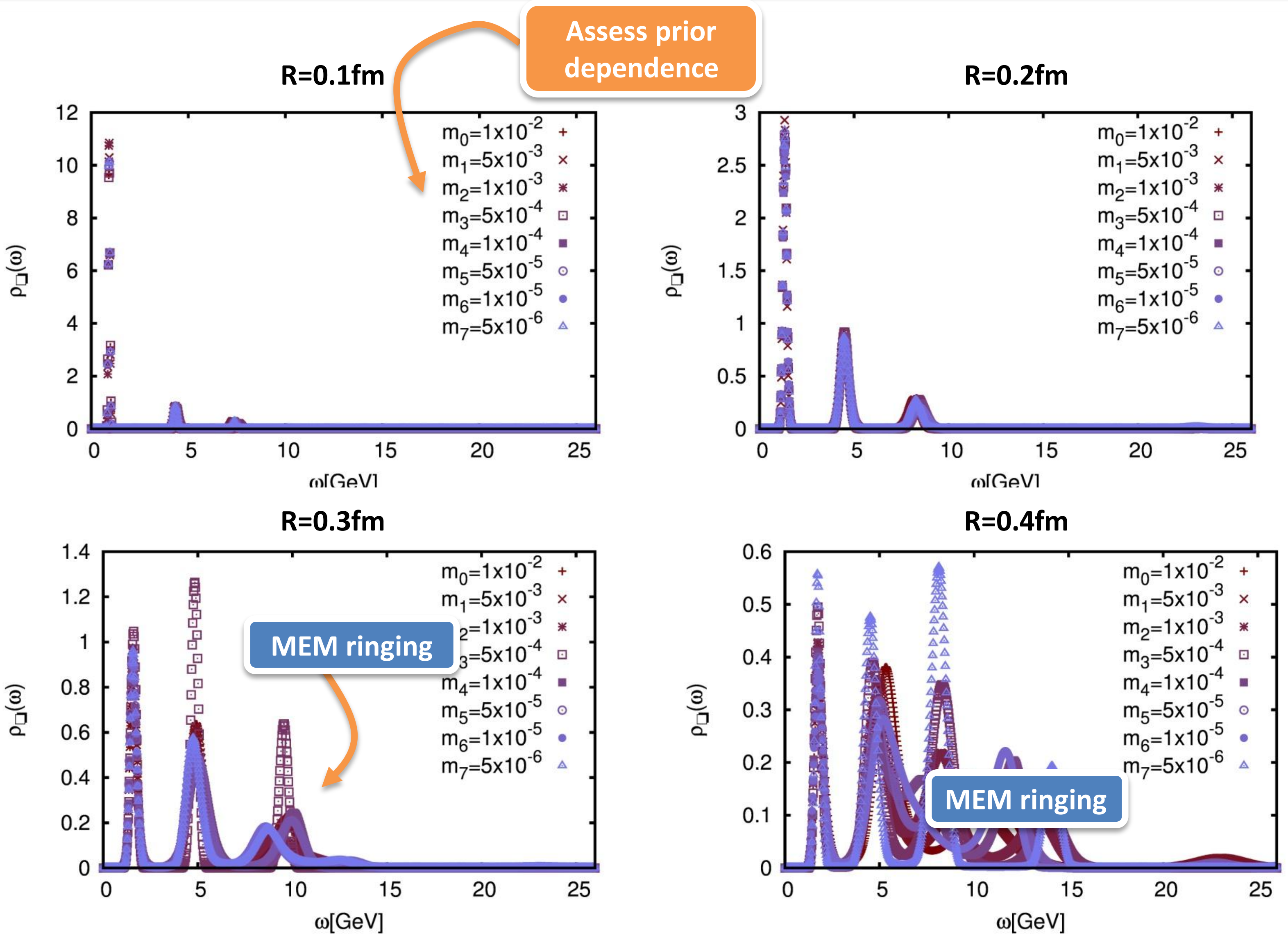

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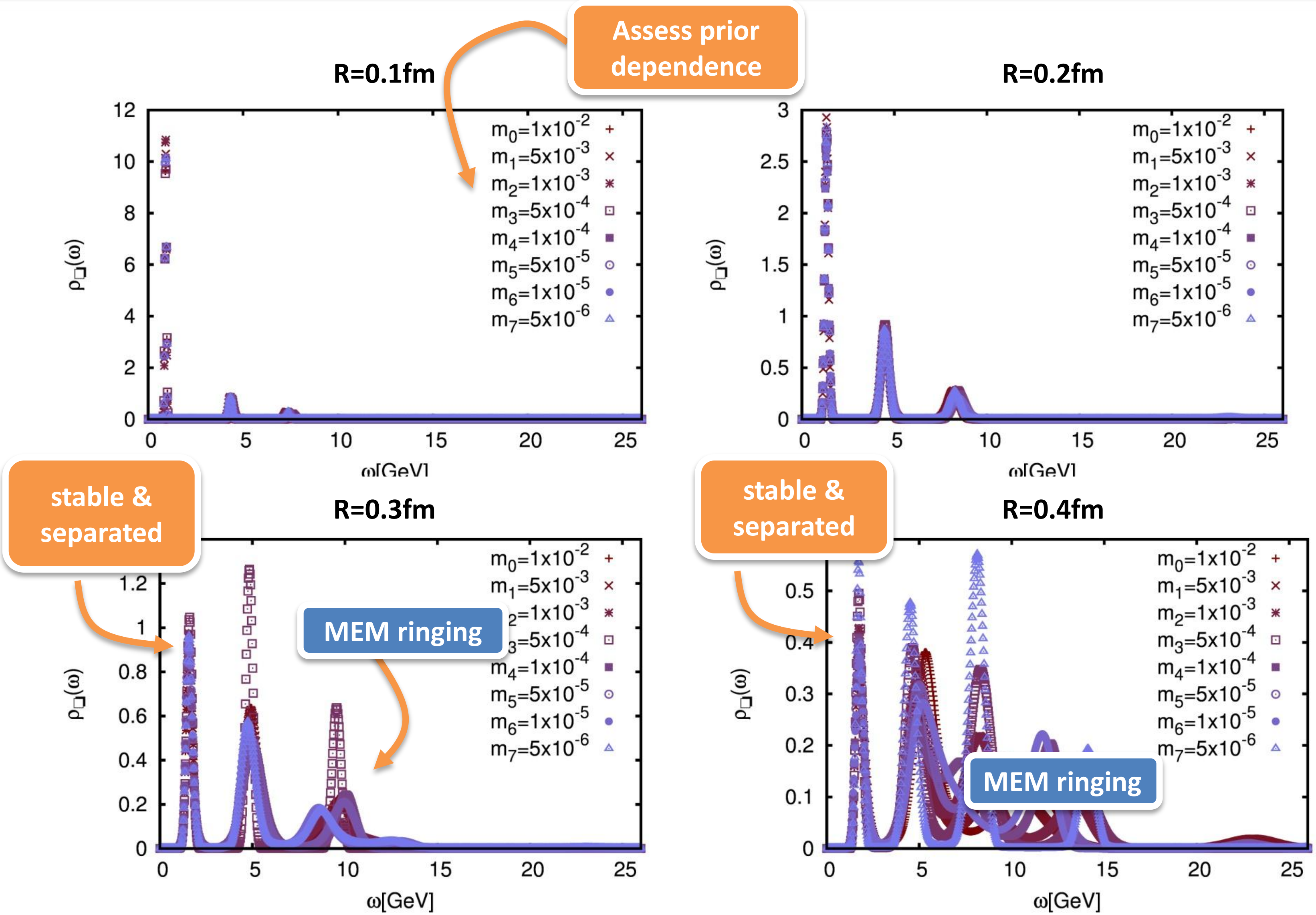

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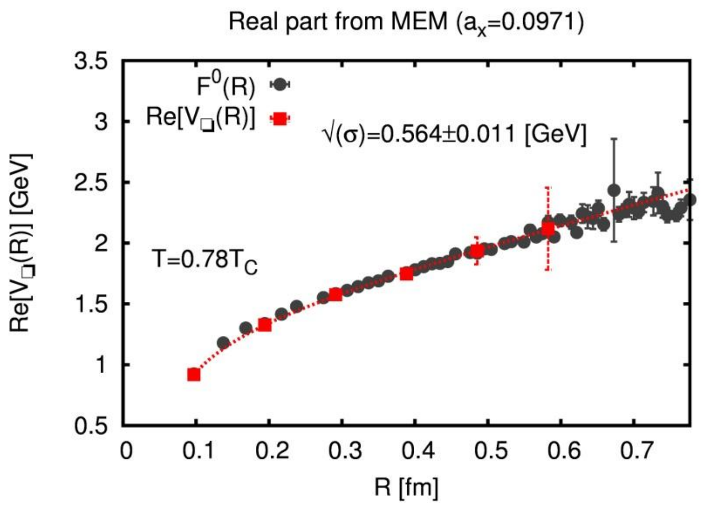

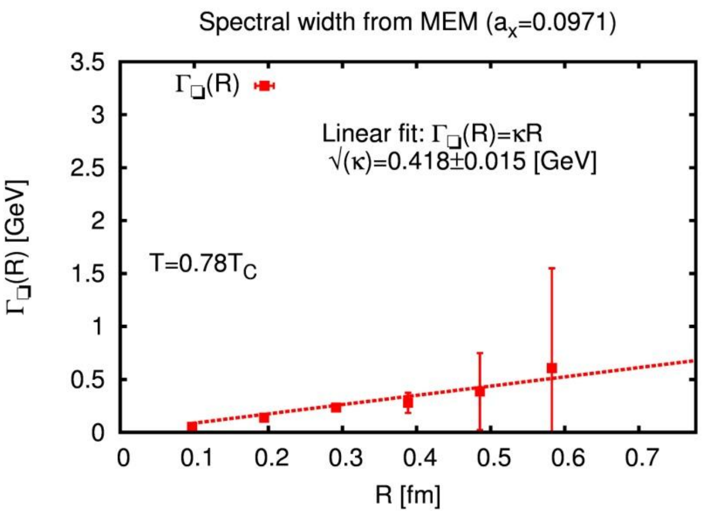

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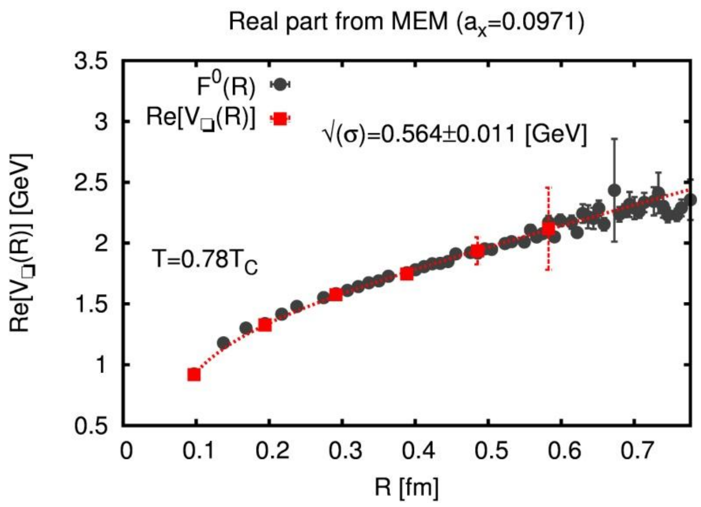
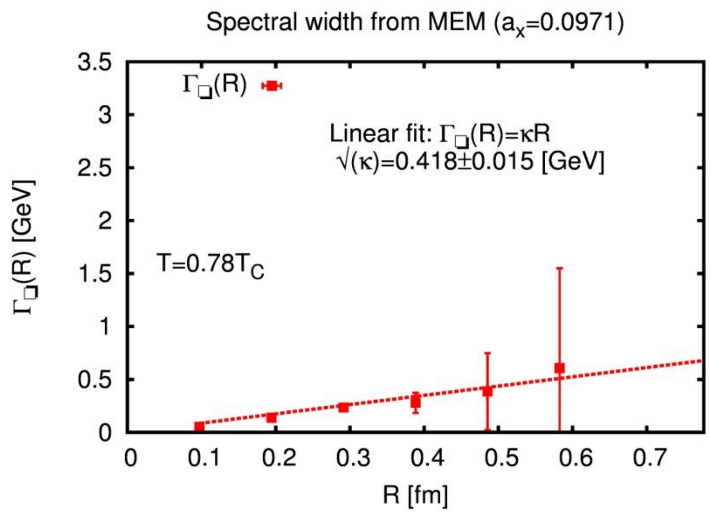
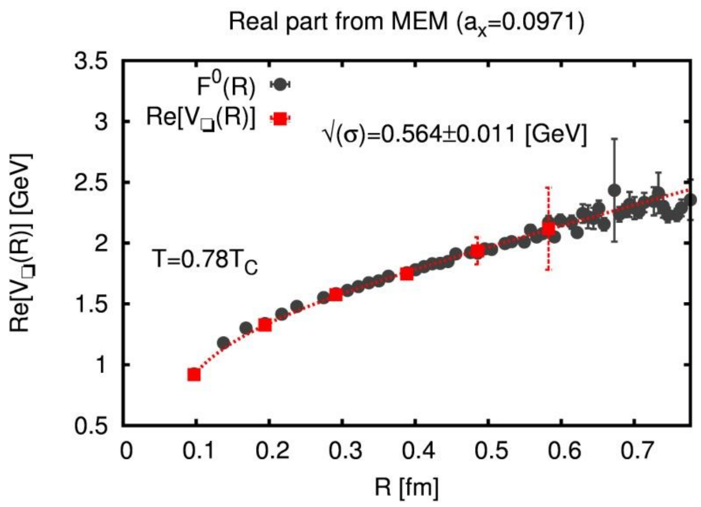
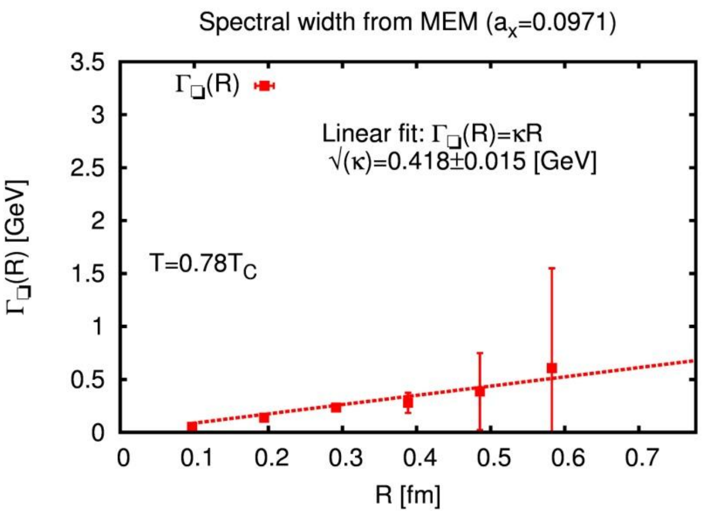




- The **real part coincides** with the color singlet free energies in Coulomb gauge

Numerical Results: T=0.78T_C





- The **real part coincides** with the color singlet free energies in Coulomb gauge
- Spectral width **consistent with zero** due to large error bars
  (Note: MEM induces artificial width)

Numerical Results: $T=1.17T_C$

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Quenched QCD Simulations

- Anisotropic Wilson Plaquette Action
- NX=20 NT=24 $\beta=6.1$ $\xi_b=3.2108$
- Box Size: 2fm Lattice Spacing: 0.1fm
- HB:OR 1:4 with 200 sweeps/readout

Maximum Entropy Method

- Singular Value Decomposition
- $N_\omega=1500$
- Prior: m_0/ω, varied over 4 orders
- 384bit precision

Numerical Results: $T=1.17T_C$

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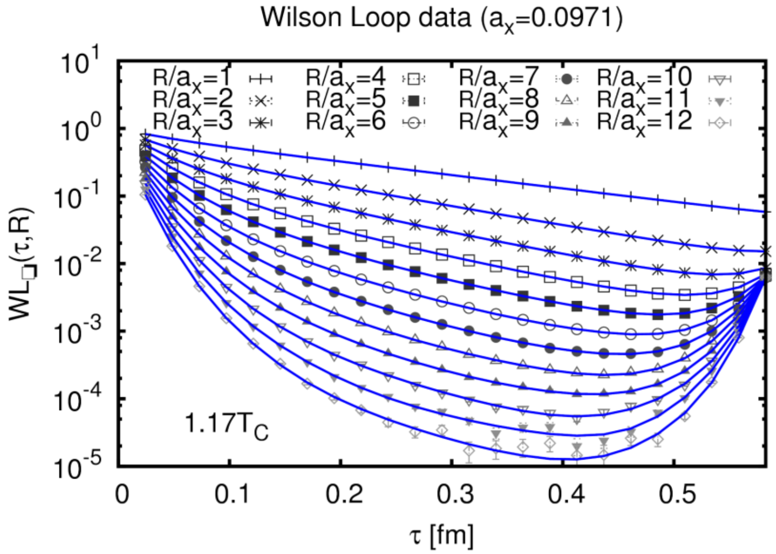

Numerical Results: T=1.17T_C

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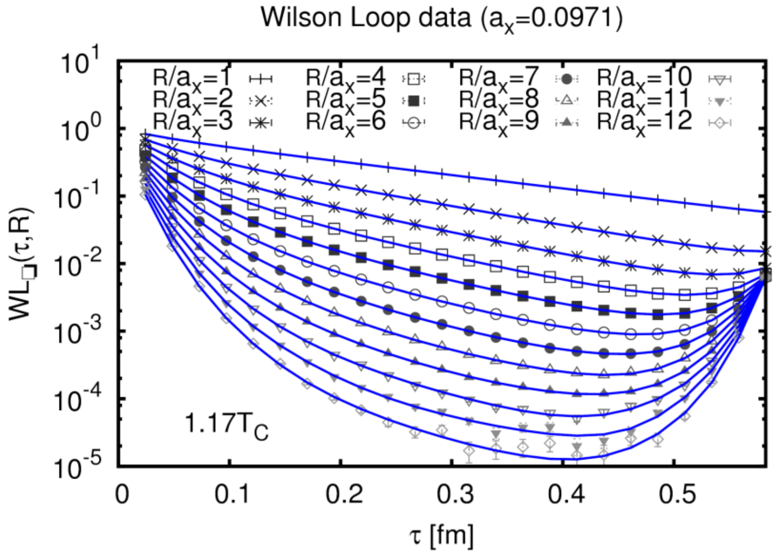
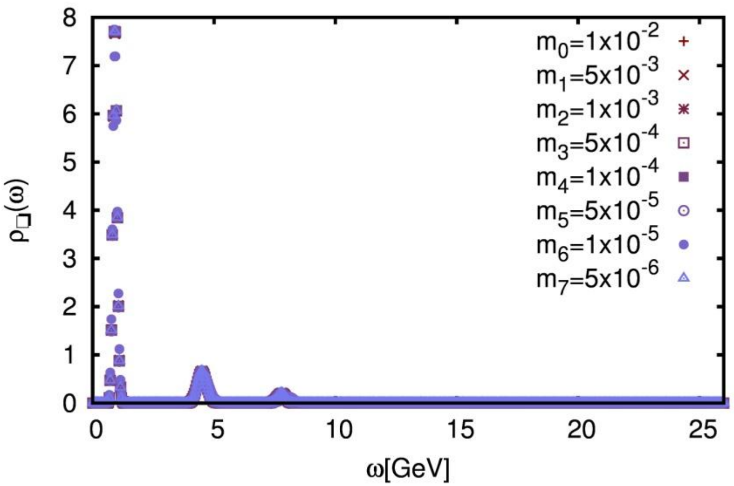


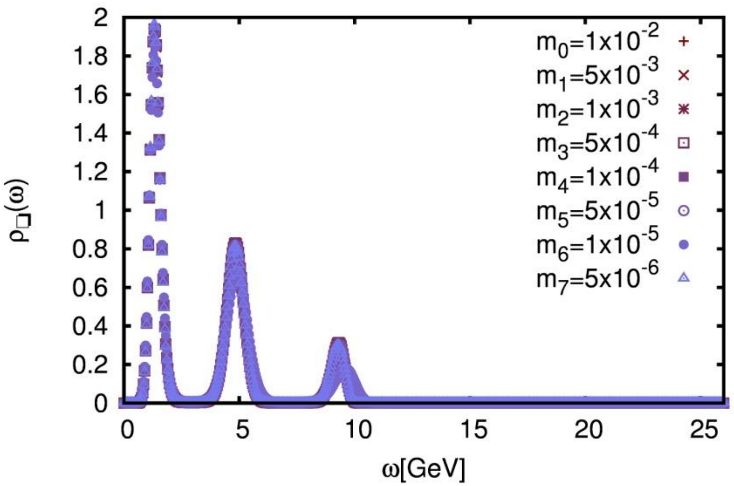
Upward trend becomes visible

Numerical Results: $T=1.17T_C$

R=0.1fm



R=0.2fm



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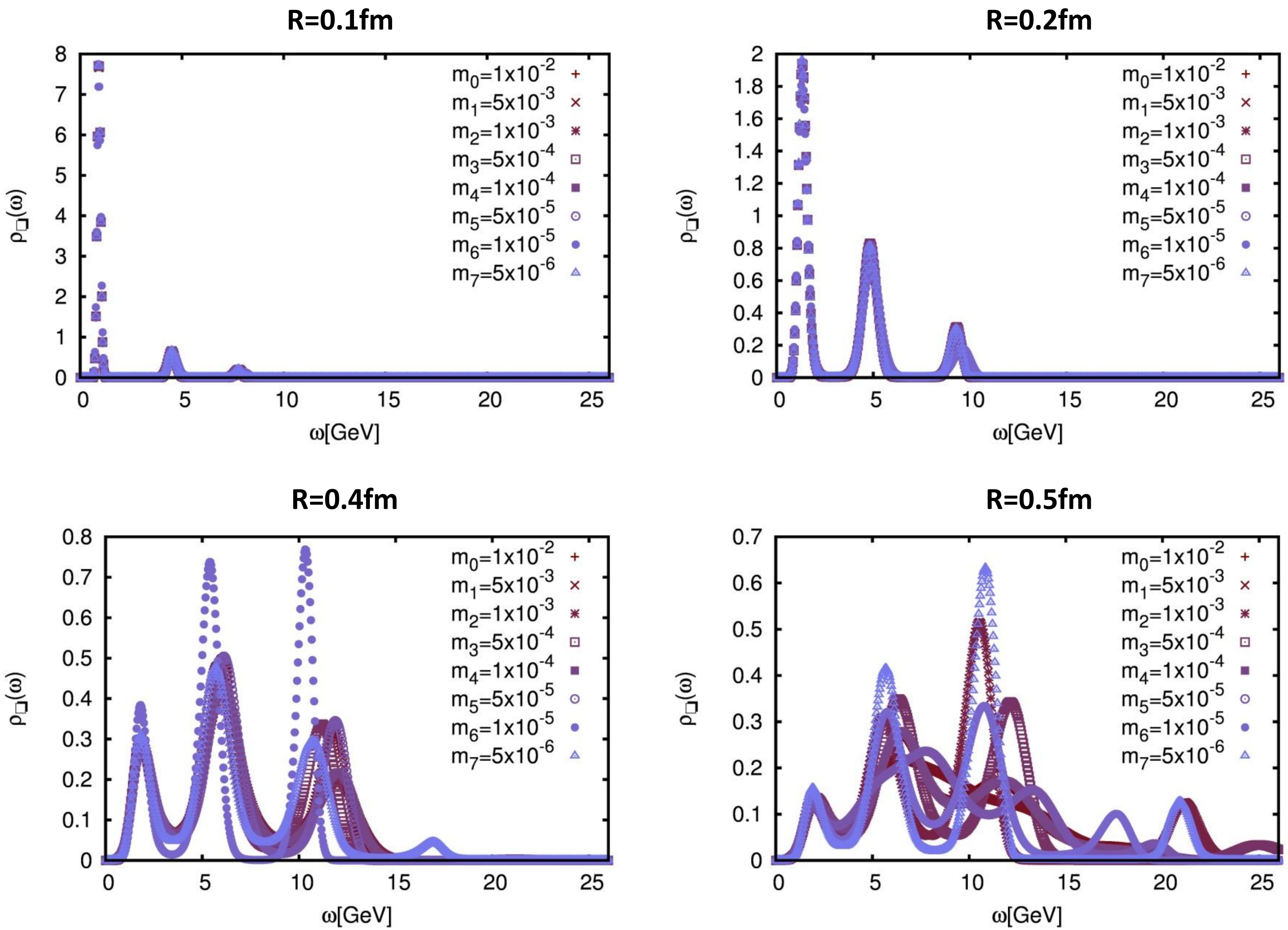

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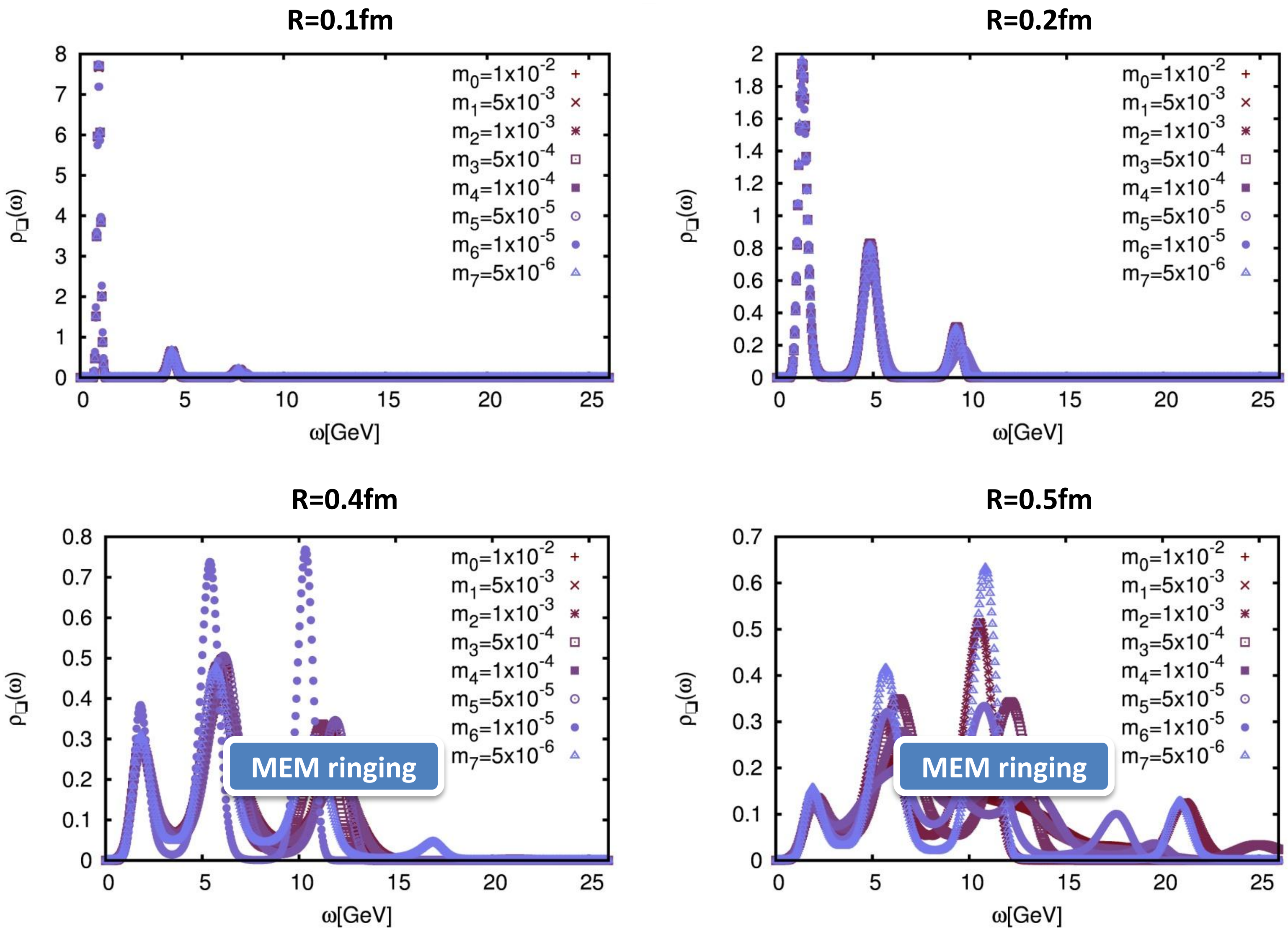

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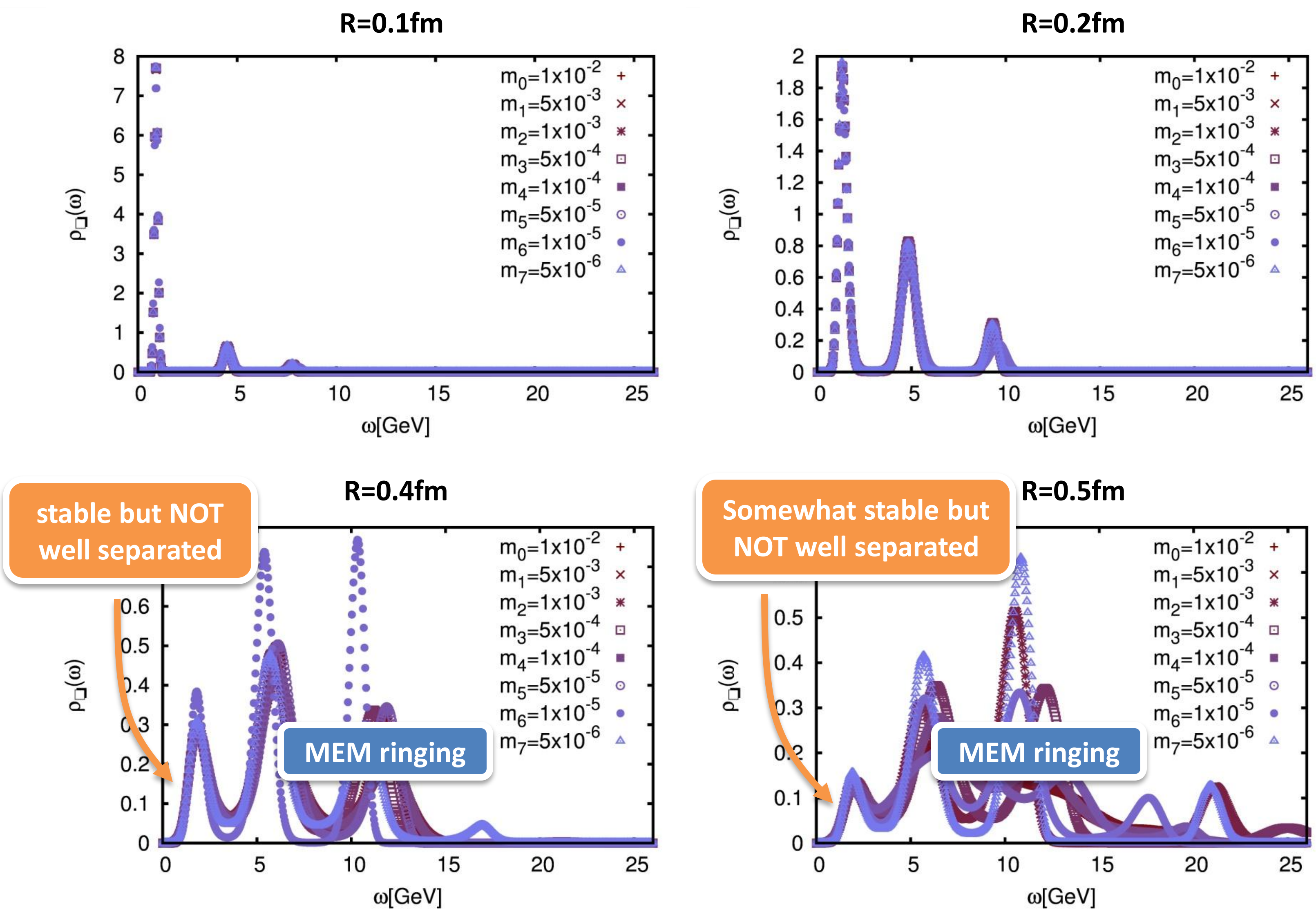

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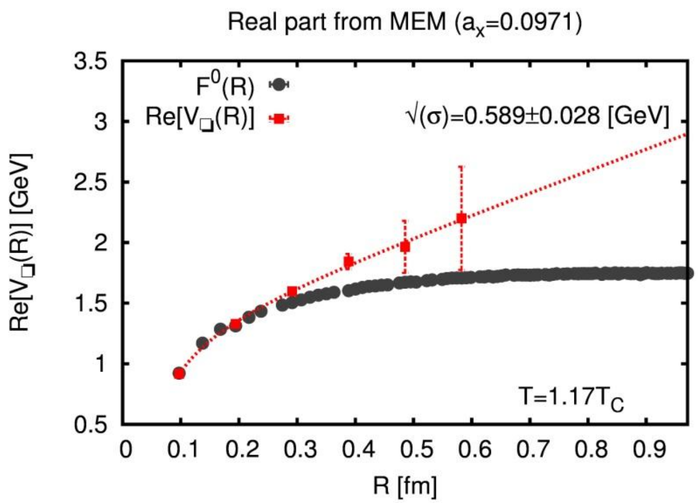

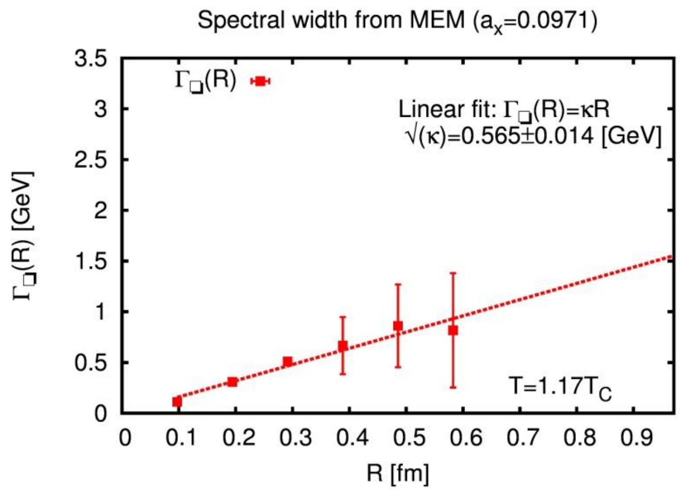

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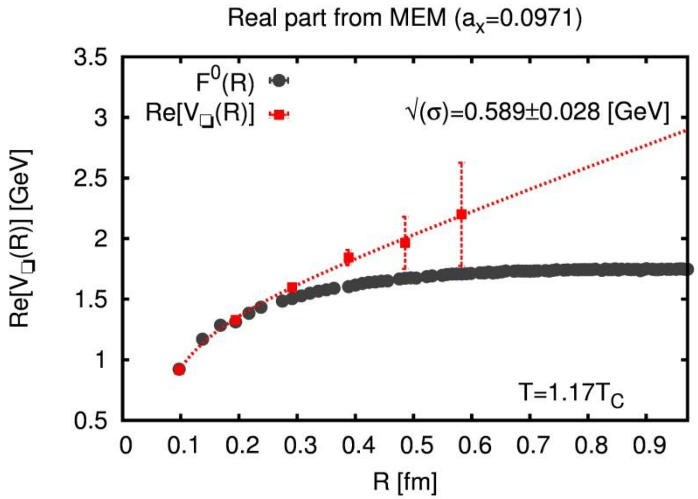

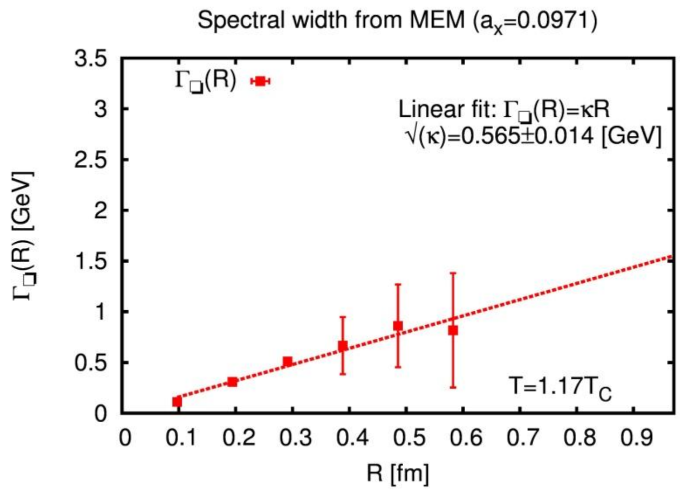

- Real part is slightly stronger than color singlet free energies but error bars are quite large.

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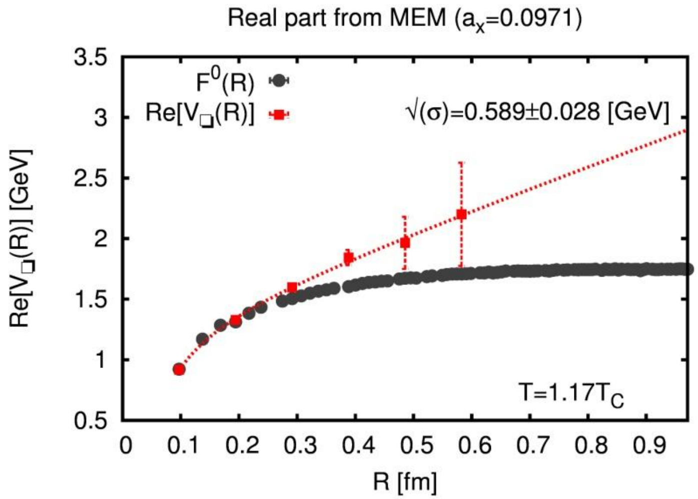
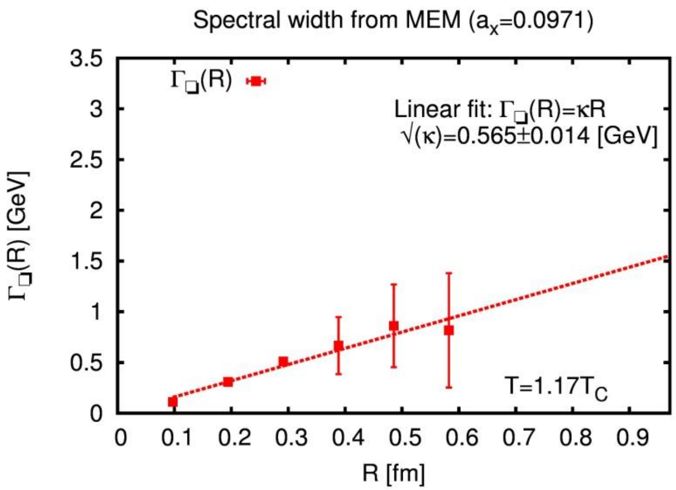
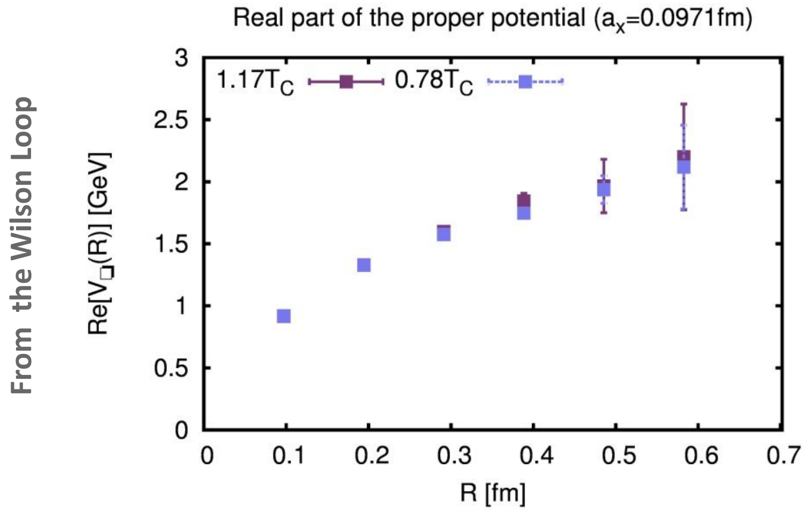


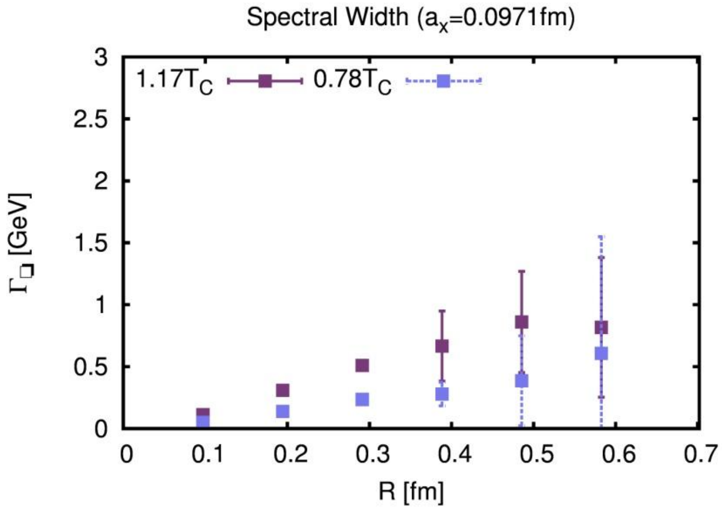


- Real part is slightly stronger than color singlet free energies but error bars are quite large.
- Spectral width is finite and larger than below T_c

Comparison at different T

- The simulations around T_C show:

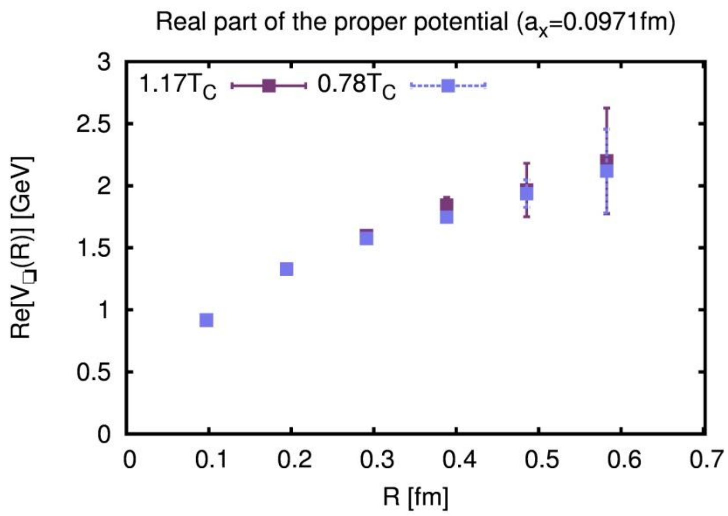


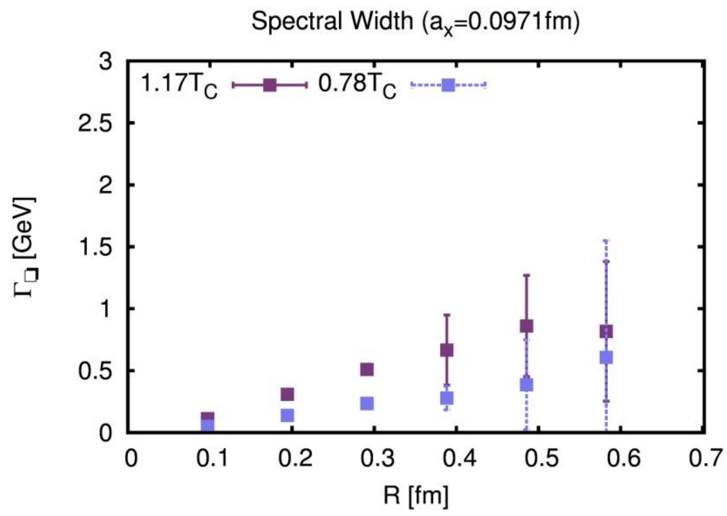


Comparison at different T

- The simulations around T_C show:

From the Wilson Loop





- **Real part** up to and around T_C **insensitive** to thermal fluctuations

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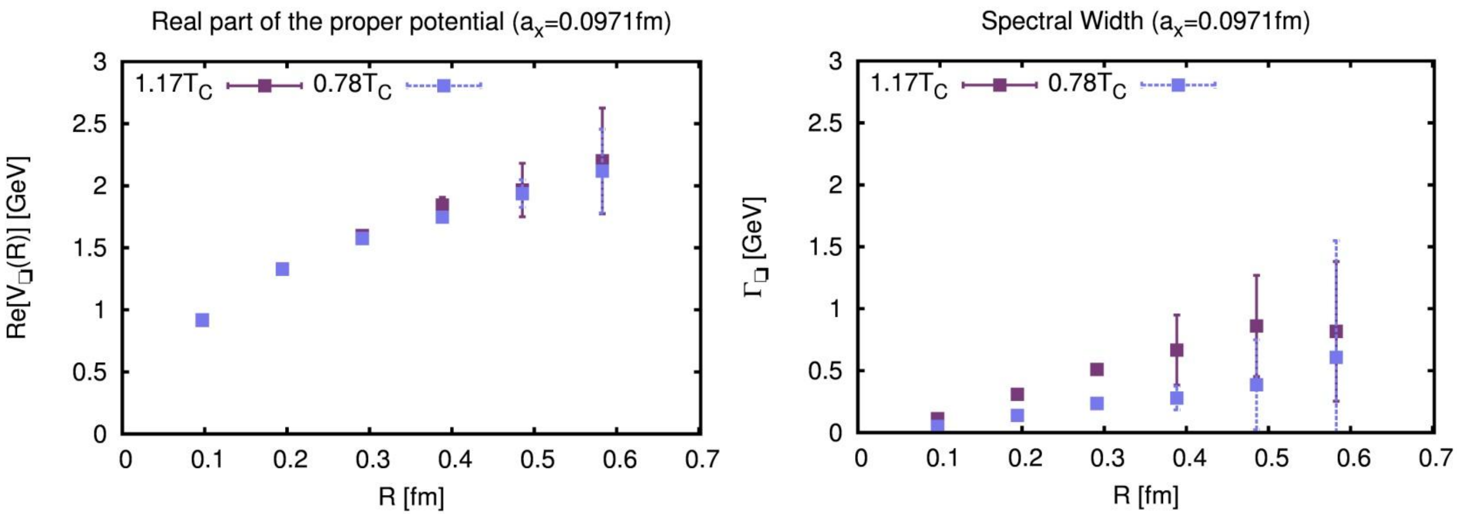

- **Real part** up to and around T_C **insensitive** to thermal fluctuations
- **Imaginary part increases** with temperature

Numerical Results T=2.33T_C

Quenched QCD Simulations

- Anisotropic Wilson Plaquette Action
- NX=20 NT=12 β=6.1 ξ_b=3.2108
- Box Size: 2fm Lattice Spacing: 0.1fm
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Maximum Entropy Method

- Singular Value Decomposition
- N_ω=1500
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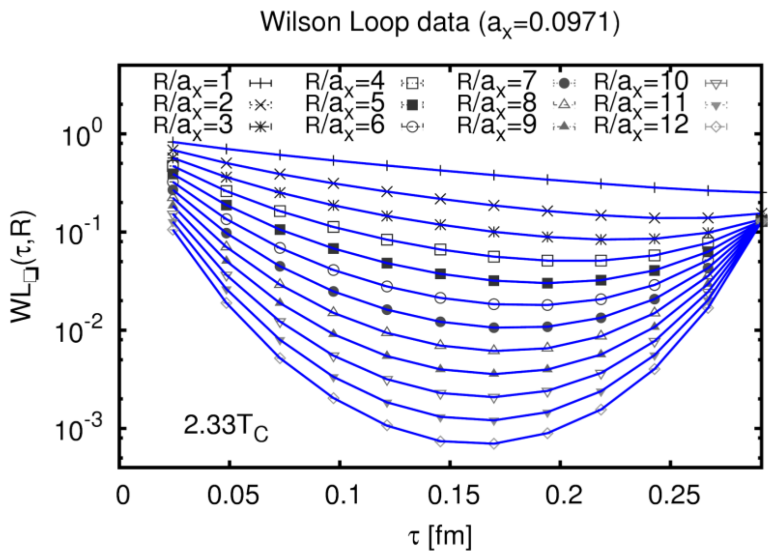

Numerical Results T=2.33T_C

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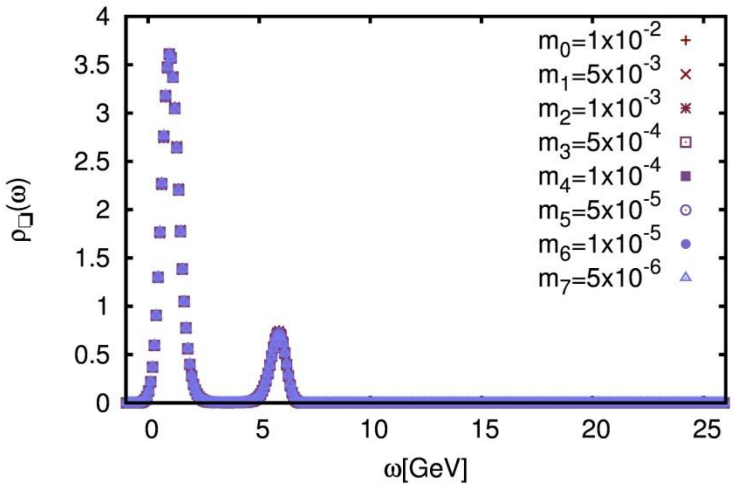
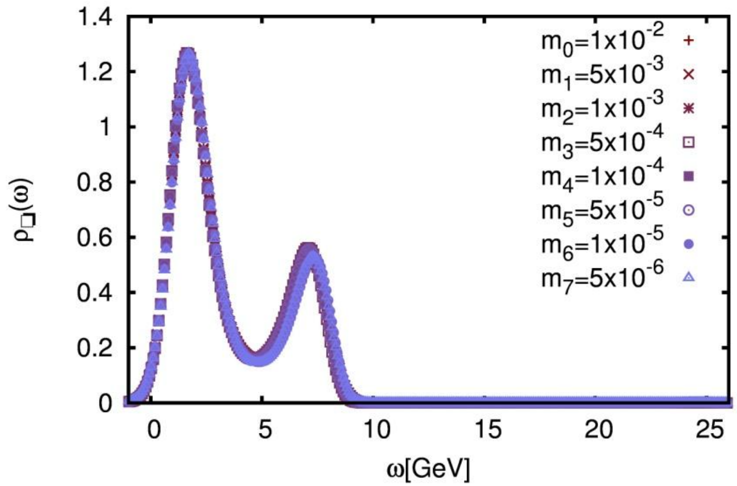
R=0.1fm



R=0.2fm



Numerical Results T=2.33T$_C$

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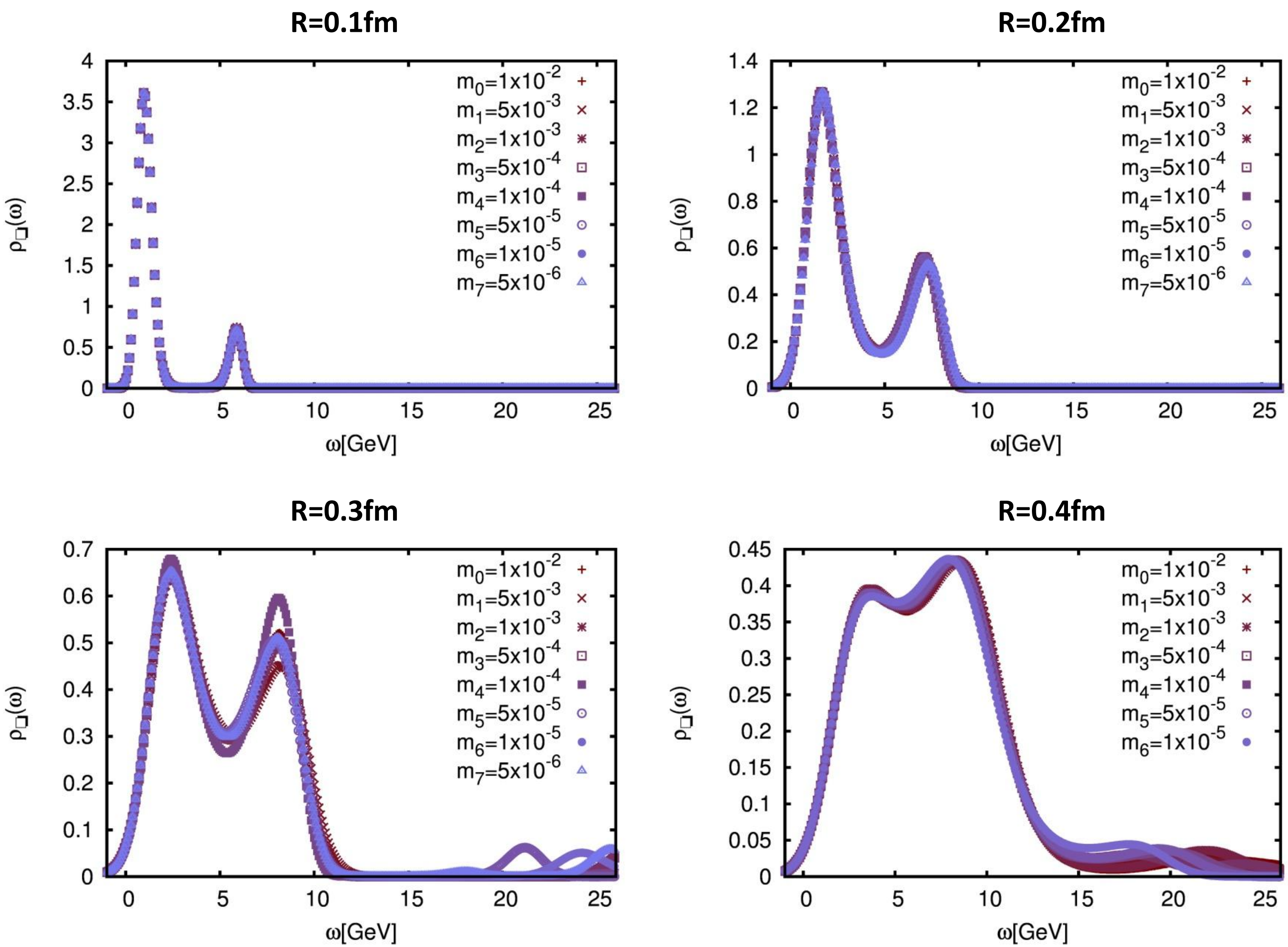

Numerical Results $T=2.33T_C$

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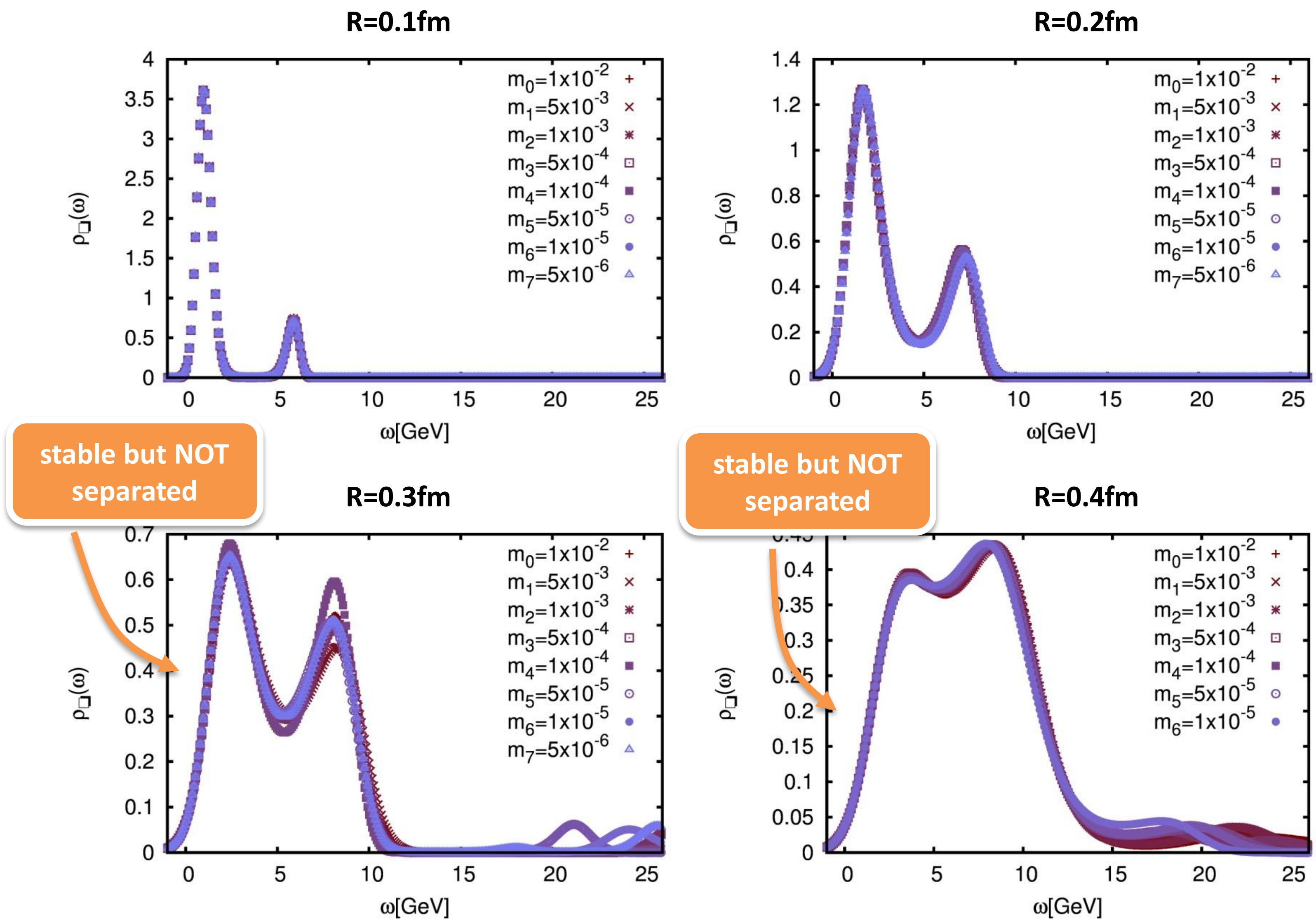

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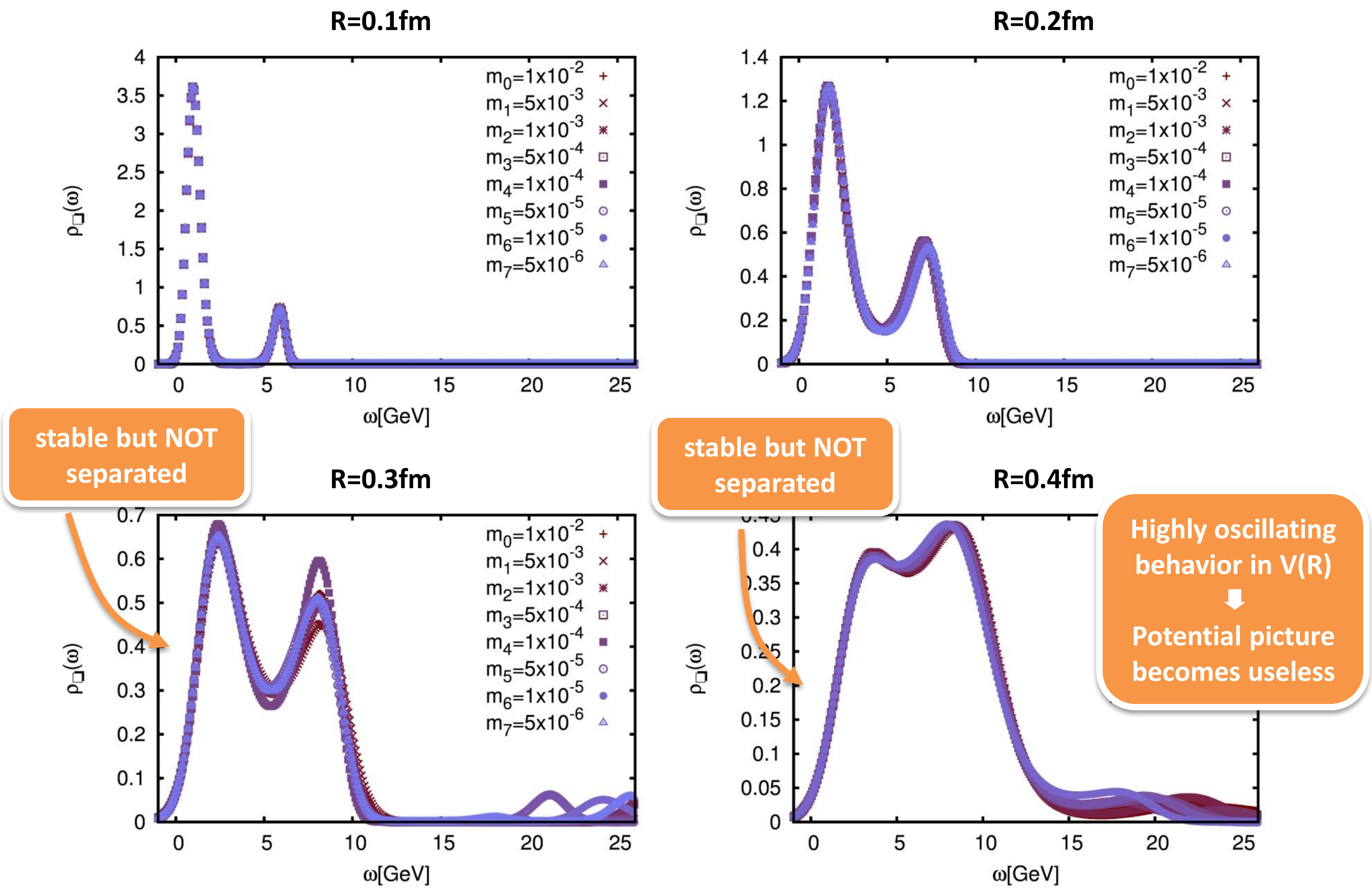

Summary & Conclusion

- Goal of this study: Contribute to the understanding of heavy Quarkonia in the QGP

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Summary & Conclusion

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Non-perturbative derivation of an effective in-medium **Schrödinger equation**

$$u(R,t) = \frac{1}{W_\square(R,t)} \int d\omega e^{-i\omega t}\ \omega\ \rho_\square(R,\omega)$$

Possibility to **check the applicability** of the potential picture

Complex Potential is obtained from the spectral function of the real-time Wilson loop

Summary & Conclusion

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東京大学 University of Tokyo

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  - Numerical Results and Discussion

    At $T<T_C$ **real part coincides with color singlet free energies**

    Up to around T the real-part appears to be **insensitive to thermal fluctuations**

    Above T_C the **applicability** of the potential picture **appears to break down**

The End

Thank you for your attention

Vielen Dank für Ihre Aufmerksamkeit