Rescattering effects in η and η' decays

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Strong interactions: from methods to structures

Bad Honnef, February 14th 2011

Schneider, BK, Ditsche, JHEP (2011); Schneider, BK, work in progress

Rescattering effects in η and η' decays – Outline

Introduction

Perturbative rescattering: $\eta \to 3\pi$ decays (1)

- Understanding the $\eta \to 3\pi^0$ Dalitz plot parameter α
- Relating charged and neutral Dalitz plot parameters

Non-perturbative rescattering: $\eta \to 3\pi$ decays (2)

- Dispersion relations for three-meson decays
- Transfer to η' decay channels

Summary

Introduction

Various reasons why final-state interactions are important

- if rescattering strong, significantly enhances decay probabilities

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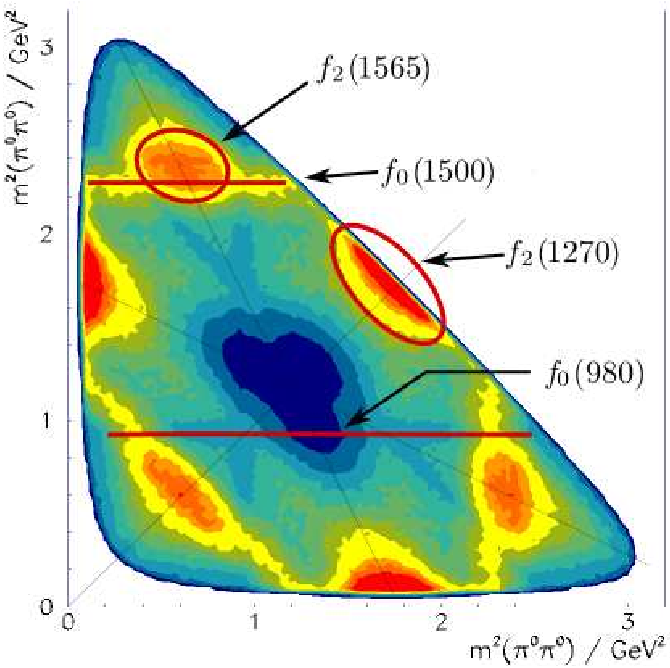
- if rescattering strong, significantly shapes decay probabilities (resonances!)



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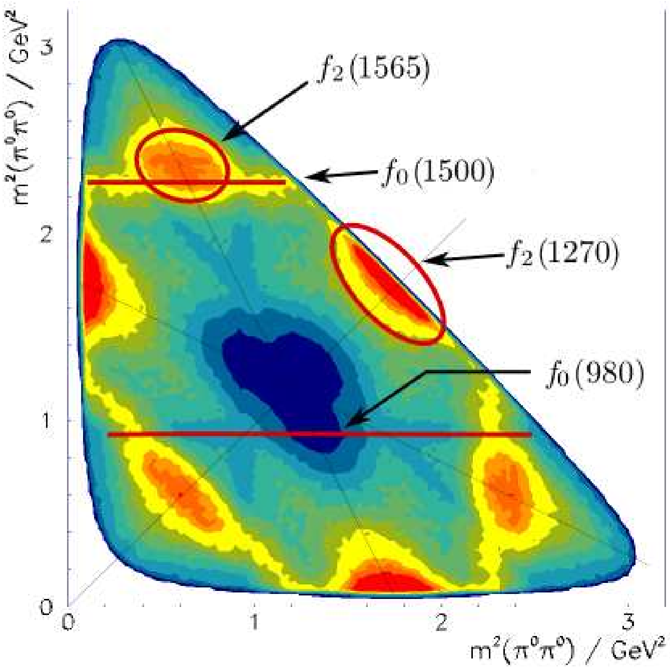

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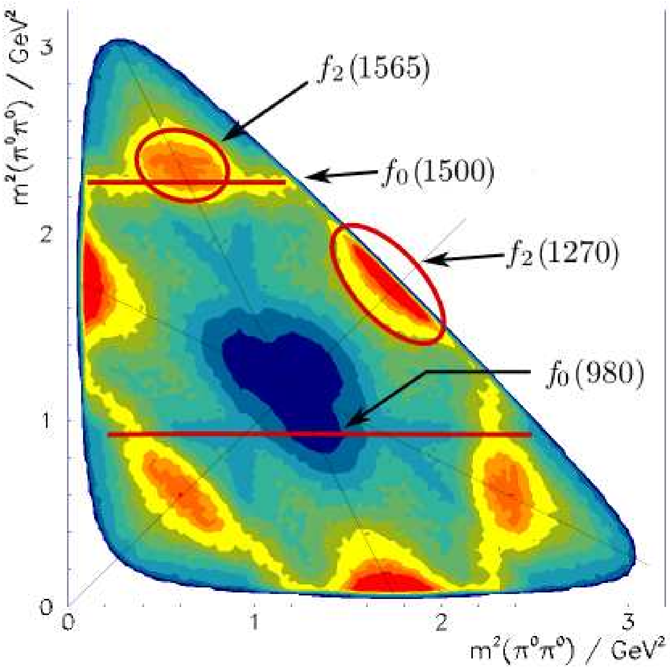
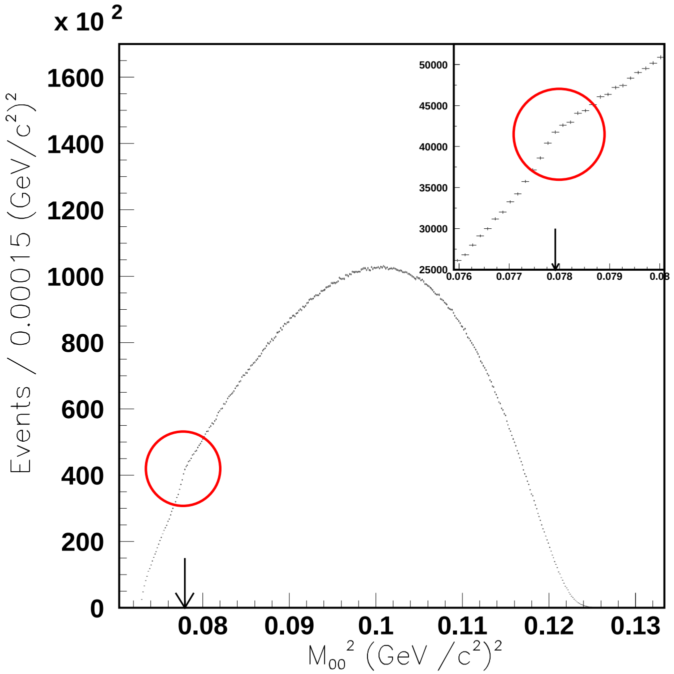
- if rescattering strong, significantly shapes decay probabilities (resonances!)
- introduce phases/imaginary parts
- new analytic features in the Dalitz plot (cusp effect in $K \to 3\pi$)





Enhancement through rescattering: $\eta \longrightarrow 3\pi$ decays

- $\eta \to 3\pi$ isospin violating; two sources in the Standard Model:

$$m_u \neq m_d \qquad\qquad e^2 \neq 0$$

- electromagnetic contribution small

Sutherland 1967

Baur, Kambor, Wyler 1996; Ditsche, BK, Meißner 2009

$$\eta \to \pi^+\pi^-\pi^0: \quad \mathcal{A}_c^{\mathsf{LO}}(s,t,u) \;=\; \frac{B(m_u-m_d)}{3\sqrt{3}F_\pi^2}\left\{1+\frac{3(s-s_0)}{M_\eta^2-M_\pi^2}\right\}$$

$$s=(p_\eta-p_{\pi^0})^2,\; t=(p_\eta-p_{\pi^+})^2,\; u=(p_\eta-p_{\pi^-})^2,\; s+t+u=M_\eta^2+3M_\pi^2 \doteq 3s_0$$

- $\Delta I = 1$ relation between charged and neutral decay amplitudes:

$$\eta \to 3\pi^0: \quad \mathcal{A}_n(s,t,u) \;=\; \mathcal{A}_c(s,t,u)+\mathcal{A}_c(t,u,s)+\mathcal{A}_c(u,s,t)$$

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$$s = (p_\eta - p_{\pi^0})^2,\ t = (p_\eta - p_{\pi^+})^2,\ u = (p_\eta - p_{\pi^-})^2,\ s+t+u = M_\eta^2 + 3M_\pi^2 \doteq 3s_0$$

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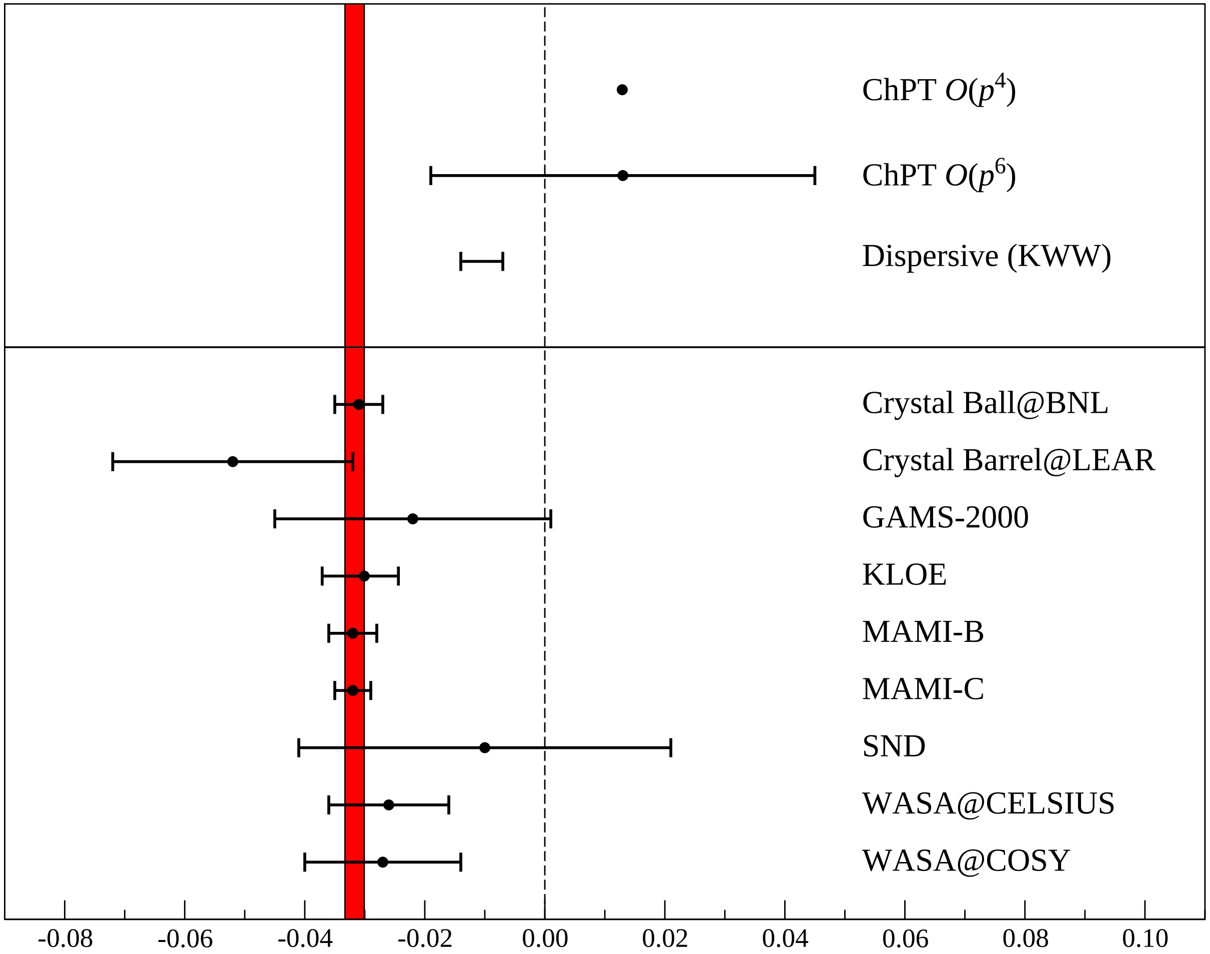
- Relevance: (potentially) clean access to $m_u - m_d$
  but: large higher-order / final-state interactions

- Chiral perturbation theory (ChPT) to 2 loops — Bijnens, Ghorbani 2007
  new dispersion-relation analyses — Colangelo, Lanz, Leutwyler, Passemar; Zdráhal, Kampf, Knecht, Novotný

- strong experimental activities — WASA-at-COSY, MAMI-B/-C, KLOE, ELSA

$\eta \longrightarrow 3\pi^0$ Dalitz plot parameter α: a puzzle

$$|\mathcal{A}_n(x,y)|^2 \;=\; |\mathcal{N}_n|^2\{1 + 2\,\alpha\, z + \ldots\} \qquad z \propto (s-s_0)^2 + (t-s_0)^2 + (u-s_0)^2$$



$10^3 \times \alpha$
$+13$
$+13 \pm 32$
$-7 \ldots -14$

PDG average:
-31.7 ± 1.6

- puzzle: why isn't two-loop ChPT closer to dispersive result?

Bijnens, Ghorbani 2007
Kambor, Wiesendanger, Wyler 1995

Precision rescattering: “non-relativistic” EFT

- theoretical tool for $\pi\pi$ scattering length extraction from cusp in $K \to 3\pi$: non-relativistic effective field theory

  Colangelo, Gasser, BK, Rusetsky 2006

  ▷ parametrise T directly in terms of scattering lengths etc.

  ▷ no quark-mass expansion of these parameters ($\leftrightarrow$ ChPT)

  ▷ retain recoil corrections $\Leftrightarrow$ correct relativistic kinematics
  only inelasticities (far outside physical region) neglected

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- combined expansion in $a_{\pi\pi}$ ($\pi\pi$ scattering lengths) and non-relativistic parameter $\epsilon \propto |\mathbf{p}_\pi|/M_\pi$

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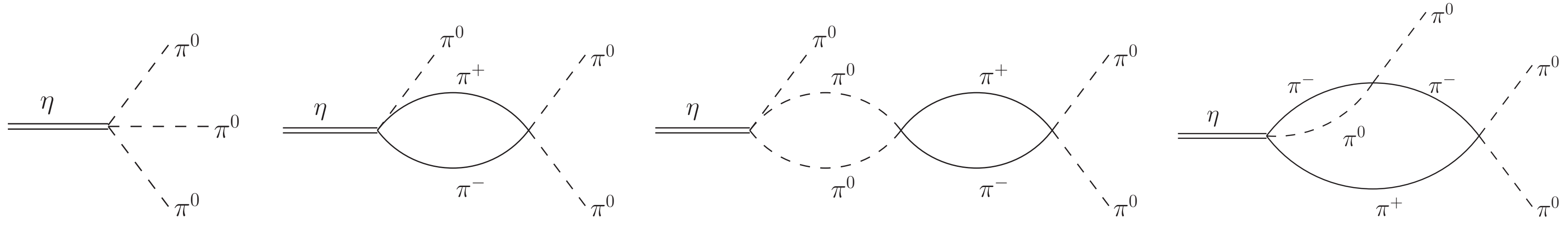
- combined expansion in $a_{\pi\pi}$ ($\pi\pi$ scattering lengths) and non-relativistic parameter $\epsilon \propto |\mathbf{p}_\pi|/M_\pi$

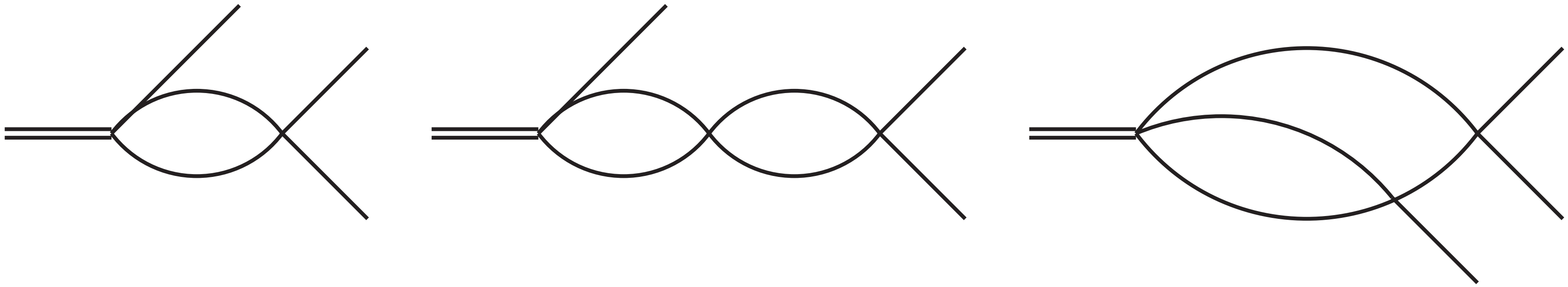
- complete to $\mathcal{O}(\epsilon^4, a_{\pi\pi}\epsilon^5, a_{\pi\pi}^2\epsilon^4)$:



- $\eta \to 3\pi$ tree couplings matched to ChPT $\mathcal{O}(p^4)$
  $\pi\pi$ from phenomenology (including isospin breaking)

Understanding α in NREFT

Schneider, BK, Ditsche 2010



$$\alpha = \big(\underbrace{+10.7}_{\text{tree}}\ \underbrace{+12.4}_{\text{1-loop}}\ \underbrace{-44.1}_{\text{2-loop}}\ \underbrace{-6.0}_{\text{higher}}\ \underbrace{-0.6}_{\text{isospin}}\big) \times 10^{-3} = (-24.6 \pm 4.9) \times 10^{-3}$$

- $\eta \to 3\pi$ tree couplings matched to ChPT $\mathcal{O}(p^4)$
- error: (1) $\pi\pi$ scattering Ananthanarayan et al. 2001 vs. Kamiński et al. 2008
  (2) estimate of higher orders ("bubble resummation")
- NREFT power counting tells us:
  - ▷ 1-loop and 2-loop both of same $\mathcal{O}(a_{\pi\pi}^2)$
  - ▷ rescattering enhanced in α: loops $\mathcal{O}(\epsilon^2)$ vs. tree $\mathcal{O}(\epsilon^4)$

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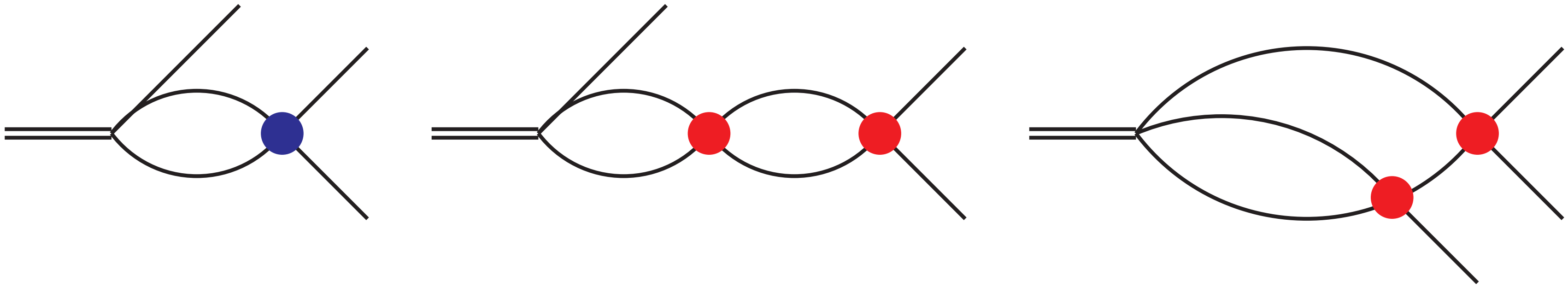

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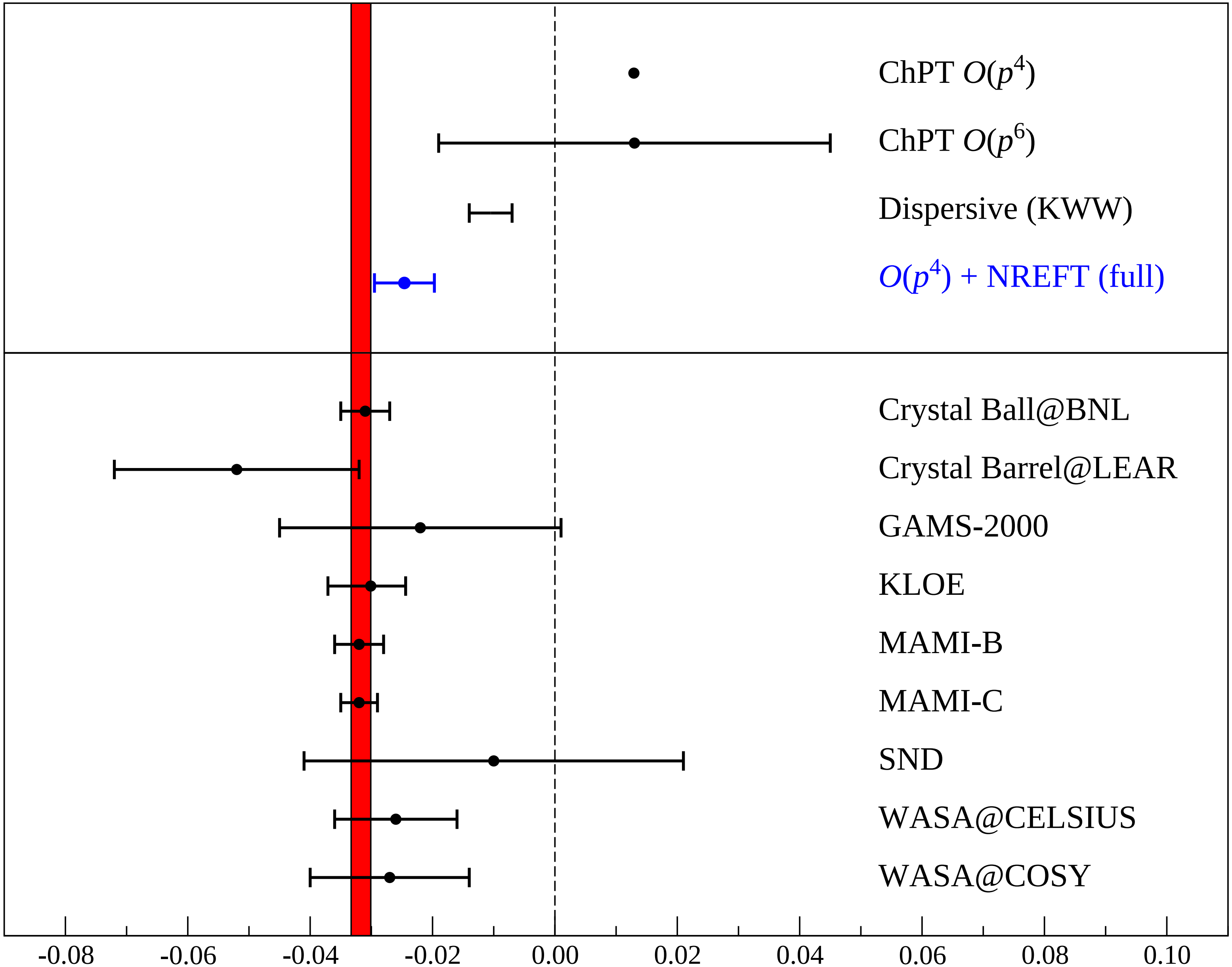
Why is the ChPT $\mathcal{O}(p^6)$ result so different, $\alpha = (+13 \pm 32) \times 10^{-3}$?

Bijnens, Ghorbani 2007

- "emulate" the chiral two-loop result:
  - ●: rescattering parameters $\mathcal{O}(p^4)$ in 1-loop graphs
  - ●: rescattering parameters $\mathcal{O}(p^2)$ in 2-loop graphs,
    e.g. $a_0^0(p^2) = 0.16$ instead of $a_0^0 = 0.22$, enters squared...
- result: find $\alpha = -1.1 \times 10^{-3}$!
  $\Rightarrow$ "weaker" rescattering at 2 loops leads to totally different result

Total result for α



Isospin relation for Dalitz plot parameters

- Dalitz plot vs. amplitude expansion: $x \propto t - u\,,\;\; y \propto s - s_0$

$$|\mathcal{A}_c|^2 = |\mathcal{N}_c|^2\{1 + ay + by^2 + dx^2 + \ldots\} \qquad |\mathcal{A}_n|^2 = |\mathcal{N}_n|^2\{1 + 2\alpha z + \ldots\}$$

$$\mathcal{A}_c \;=\; \mathcal{N}_c \;\{1 + \bar{a}y + \bar{b}y^2 + \bar{d}x^2 + \ldots\} \qquad \mathcal{A}_n \;=\; \mathcal{N}_n \;\{1 + \bar{\alpha}z + \ldots\}$$

$$a = 2\,\mathrm{Re}\,\bar{a}\,, \quad b = |\bar{a}|^2 + 2\,\mathrm{Re}\,\bar{b}\,, \quad d = 2\,\mathrm{Re}\,\bar{d}\,, \quad \alpha = \mathrm{Re}\,\bar{\alpha}$$

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- isospin relation between neutral and charged parameters:

$$\bar{\alpha} = \frac{1}{2}\left(\bar{b} + \bar{d}\right) \;\Rightarrow\; \alpha = \frac{1}{4}\left(b + d - \frac{a^2}{4} - (\mathrm{Im}\,\bar{a})^2\right) < \frac{1}{4}\left(b + d - \frac{a^2}{4}\right)$$

Bijnens, Ghorbani 2007

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$$\alpha = \frac{1}{4}\left(b + d - \frac{a^2}{4}\right) - \zeta_1(1 + \zeta_2 a)^2\,, \ \zeta_1 = 0.050 \pm 0.005\,, \ \zeta_2 = 0.225 \pm 0.003$$

$\zeta_{1/2}$ determined by $\pi\pi$ phases

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$\zeta_{1/2}$ determined by $\pi\pi$ phases — Schneider, BK, Ditsche 2010

- use precise KLOE data on a, b, d as input:

$$\alpha^{\text{theo}}_{\text{KLOE}} = -0.062\pm0.003_{\text{stat}}{}^{+0.004}_{-0.006}{}_{\text{syst}}\pm0.003_{\pi\pi}$$

$$\alpha^{\text{exp}}_{\text{KLOE}} = -0.030\pm0.004_{\text{stat}}{}^{+0.002}_{-0.004}{}_{\text{syst}}$$

} significant tension!

Isospin relation for Dalitz plot parameters

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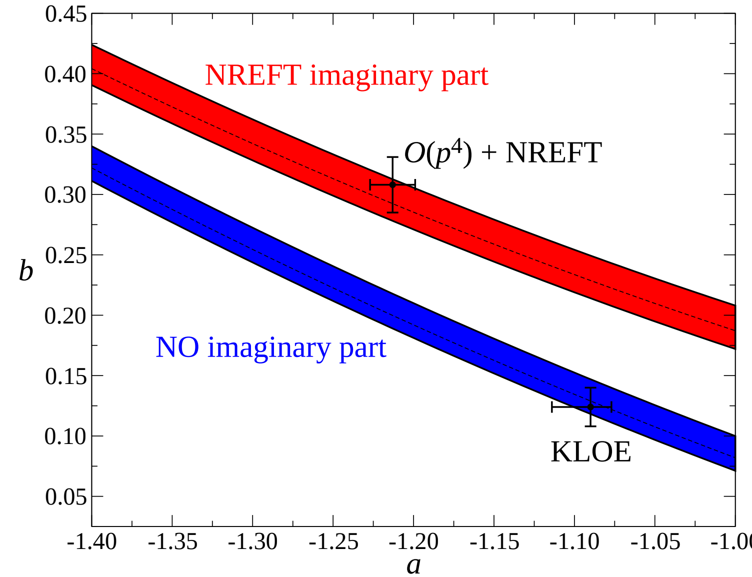
displayed as constraint in $a-b$ plane:



Dispersion relations for $\eta \longrightarrow 3\pi$: essentials

Fundamentals

- aim: resum $\pi\pi$ rescattering to all orders
- use modern high-precision parametrizations of $\pi\pi$ scattering
  Ananthanarayan et al. 2001, Kamiński et al. 2008
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- $\mathcal{M}(s,t,u) \propto \mathcal{A}_c(s,t,u)$ can be decomposed according to

$$\mathcal{M}(s,t,u) = \mathcal{M}_0(s)+(s-t)\mathcal{M}_1(u)+(s-u)\mathcal{M}_1(t)+\mathcal{M}_2(t)+\mathcal{M}_2(u)-\frac{2}{3}\mathcal{M}_2(s)$$

$\mathcal{M}_I(s)$ functions of one variable with only a right-hand cut

Stern, Sazdjian, Fuchs 1993; Anisovich, Leutwyler 1998

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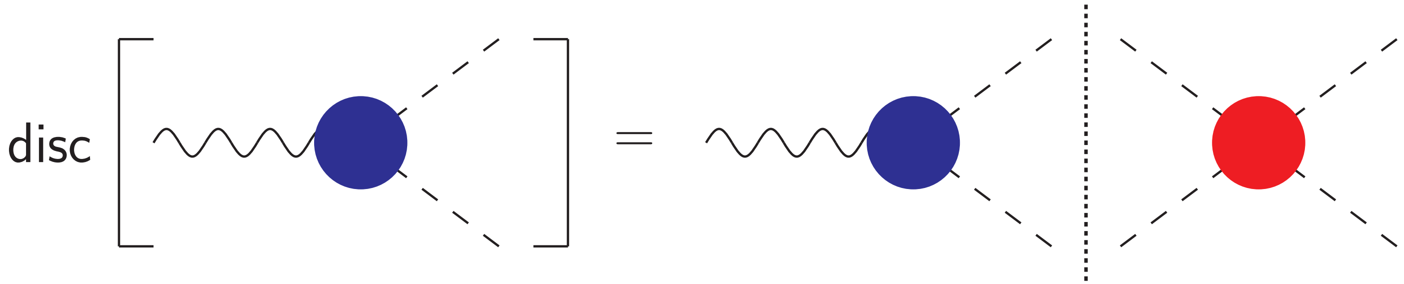
$\mathcal{M}_I(s)$ functions of one variable with only a right-hand cut

Stern, Sazdjian, Fuchs 1993; Anisovich, Leutwyler 1998

- decomposition exact only if $l \geq 2$ partial waves are real
  $\hat{=}$ $\mathcal{O}(p^8)$ or 3 loops in chiral counting

From unitarity to integral equations: form factor

- just two particles in final state (form factor); from unitarity:



$$\text{disc}\, F_I(s) \quad = \quad F_I(s) \times \theta(s - 4\,M_\pi^2) \times \sin\,\delta_I(s)\, e^{i\delta_I(s)}$$

$\Rightarrow$ Watson's final-state theorem — Watson 1954

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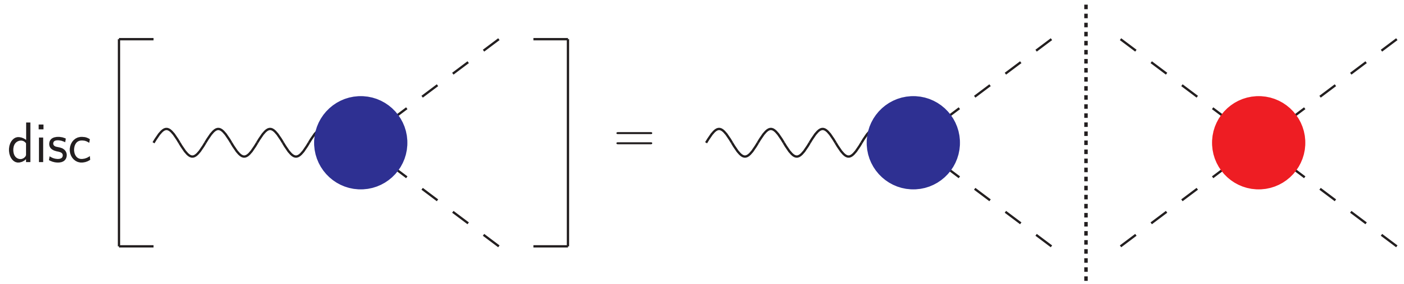

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- solution to this homogeneous integral equation known:

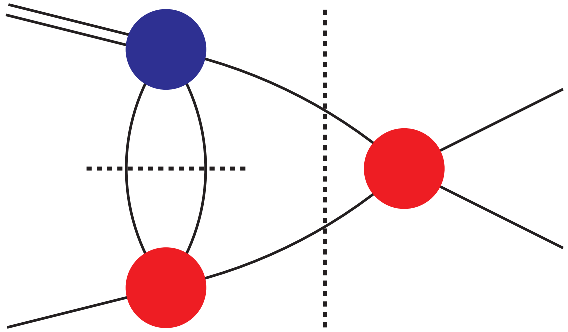
$$F_I(s) = P_I(s)\Omega_I(s)\,, \quad \Omega_I(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s)} \right\}$$

$P_I(s)$ polynomial, $\Omega_I(s)$ Omnès function — Omnès 1958

completely given in terms of phase shift $\delta_I(s)$

From unitarity to integral equations: inhomogeneities

- more complicated unitarity relation for 4-point functions:



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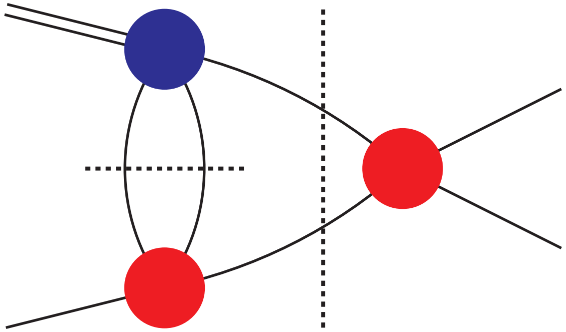

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- inhomogeneities $\hat{\mathcal{M}}_I(s)$: angular averages over the $\mathcal{M}_I(s)$: e.g.

$$\hat{\mathcal{M}}_0 = \tfrac{2}{3}\langle\mathcal{M}_0\rangle + \tfrac{20}{9}\langle\mathcal{M}_2\rangle + 2(s-s_0)\langle\mathcal{M}_1\rangle + \tfrac{2}{3}\kappa\langle z\mathcal{M}_1\rangle$$

$$\langle z^n f\rangle(s) = \tfrac{1}{2}\int_{-1}^{1} dz\, z^n f\left(\tfrac{1}{2}(3s_0 - s + z\kappa(s))\right)$$

$$\kappa(s) = \sqrt{1 - \frac{4\,M_\pi^2}{s}} \times \sqrt{(s-(M_\eta+M_\pi)^2)(s-(M_\eta-M_\pi)^2)}$$

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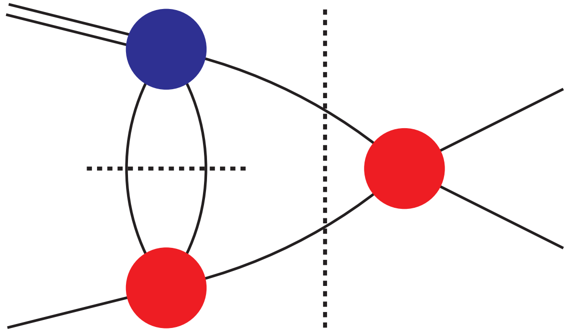

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- allows for cross-channel scattering between s-, t-, and u-channel
- $\kappa(s)$ generates complex analytic structure (3-particle cuts)

From unitarity to integral equations: solution

- integral equations including the inhomogeneities $\hat{\mathcal{M}}_I$:

$$\mathcal{M}_0(s) = \Omega_0(s)\left\{\alpha_0 + \beta_0\, s + \gamma_0\, s^2 + \frac{s^3}{\pi}\int\limits_{4M_\pi^2}^{\infty} \frac{ds'}{s'^3}\frac{\sin\delta_0(s')\hat{\mathcal{M}}_0(s')}{|\Omega_0(s')|(s'-s-i\epsilon)}\right\}$$

+ 2 similar equations for $\mathcal{M}_{1,2}(s)$

Khuri, Treiman 1960; Aitchison 1977; Anisovich, Leutwyler 1998

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- 4 subtraction constants (one more in $\mathcal{M}_1$) need to be fixed:
  - ▷ matching to $\mathcal{O}(p^4)$-ChPT at the Adler zero ("m_s-safe[r]")
  - ▷ matching to experimental data

From unitarity to integral equations: solution

- integral equations including the inhomogeneities $\hat{\mathcal{M}}_I$:

$$\mathcal{M}_0(s) = \Omega_0(s)\left\{\alpha_0 + \beta_0\, s + \gamma_0\, s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0(s')\hat{\mathcal{M}}_0(s')}{|\Omega_0(s')|(s'-s-i\epsilon)}\right\}$$

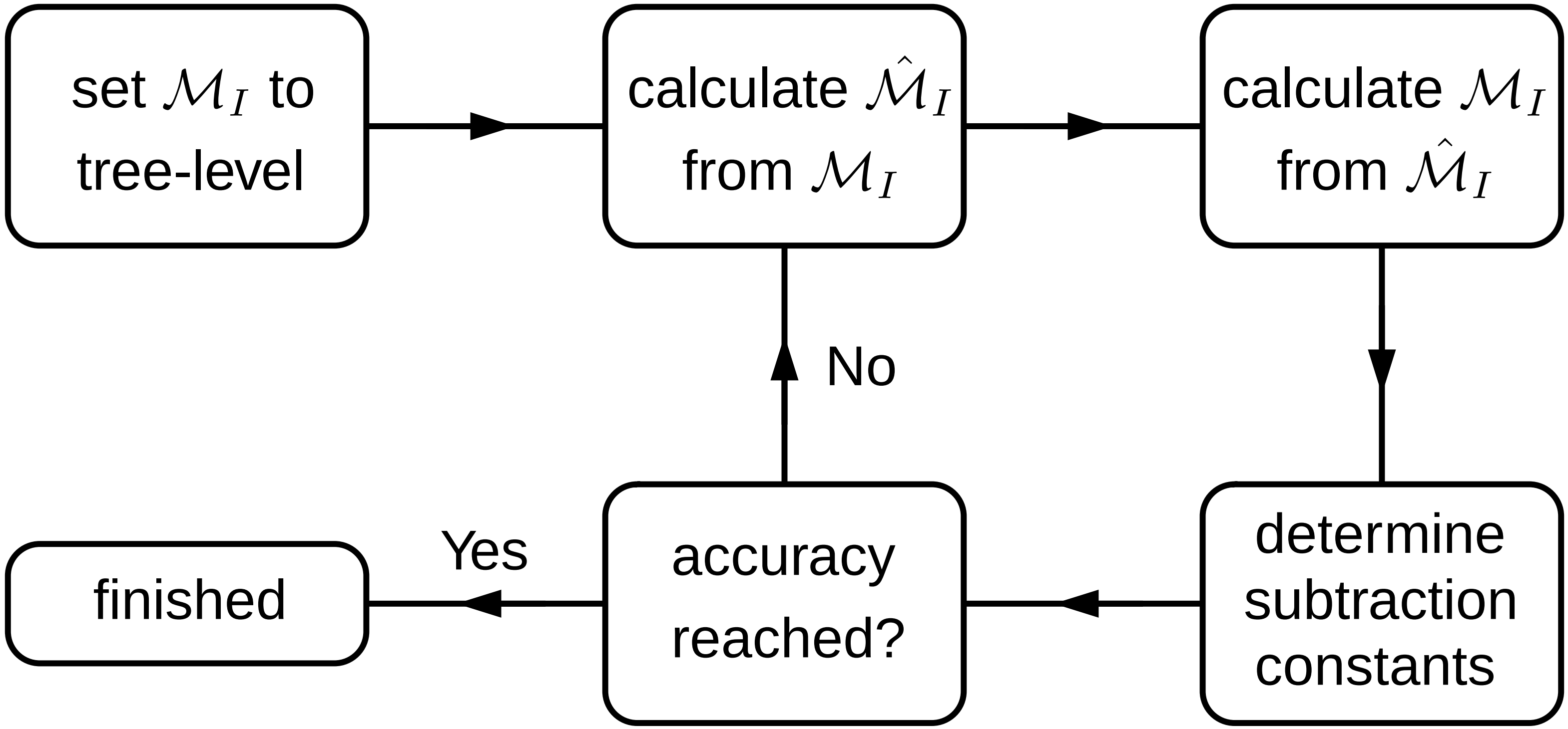
+ 2 similar equations for $\mathcal{M}_{1,2}(s)$

Khuri, Treiman 1960; Aitchison 1977; Anisovich, Leutwyler 1998

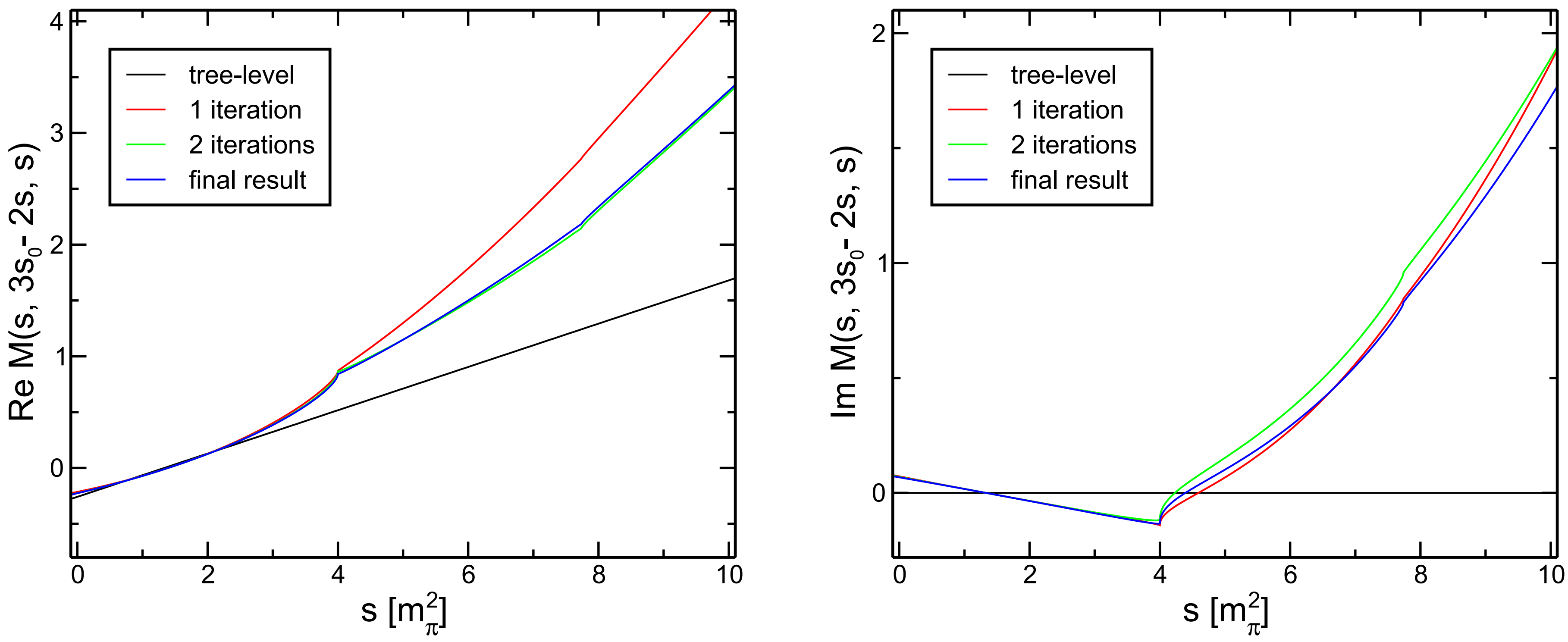
- 4 subtraction constants (one more in $\mathcal{M}_1$) need to be fixed:
  - ▷ matching to $\mathcal{O}(p^4)$-ChPT at the Adler zero ("m_s-safe[r]")
  - ▷ matching to experimental data

- solve these equations iteratively by a numerical procedure

Iterative procedure for solving integral equations



Numerical results: $\mathcal{M}(s, 3s_0 - 2s, s)$



Colangelo, Lanz, Leutwyler, Passemar 2010 (preliminary)

- fast convergence: real part almost indistinguishable from final result after 2 iterations

Hadronic η' decays

$\eta' \to \eta\pi\pi$

- one of the very few channels where $\pi\eta$ scattering can be studied
- neutral channel shows cusp effect BK, Schneider 2009
- expect short-term increase in statistical data base from BES-III, WASA@COSY, ELSA, CB@MAMI-C

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$\eta' \to 3\pi$

- again isospin-violating decay
- two possible mechanisms: direct isospin breaking $\eta' \to 3\pi$ *and* $\eta' \to \eta\pi\pi$ + isospin-breaking $\eta\pi \to \pi\pi$ rescattering
- larger phase space: ρ resonance in the decay region; inelasticities more important? how important are crossed-channel effects in the context of resonances?

Integral equations for $\eta' \longrightarrow \eta\pi\pi$

- $\eta' \to \eta\pi\pi$ amplitude decomposed into S- and P-waves:

$$\mathcal{M}(s,t,u) = \mathcal{M}_0^{\pi\pi}(s) + \mathcal{M}_0^{\pi\eta}(t) + \big\{(s-u)t + \Delta_{\eta'\pi}\Delta_{\eta\pi}\big\}\mathcal{M}_1^{\pi\eta}(t) + (t \leftrightarrow u)$$

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$$\mathcal{M}_0^{\pi\eta}(t) = \Omega_0^{\pi\eta}(t)\frac{t^2}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'^2}\frac{\sin\delta_0^{\pi\eta}(t')\hat{\mathcal{M}}_0^{\pi\eta}(t')}{|\Omega_0^{\pi\eta}(t')|(t'-t-i\epsilon)}\,, \quad t_0 = (M_\eta + M_\pi)^2$$

$$\mathcal{M}_1^{\pi\eta}(t) = \Omega_1^{\pi\eta}(t)\frac{1}{\pi}\int_{t_0}^{\infty} dt'\,\frac{\sin\delta_1^{\pi\eta}(t')\hat{\mathcal{M}}_1^{\pi\eta}(t')}{|\Omega_1^{\pi\eta}(t')|(t'-t-i\epsilon)}$$

- maybe neglect P-wave discontinuity: "exotic", real up to 3 loops
- 3 constants to be fixed; numerical implementation underway

Challenges in $\eta' \longrightarrow \eta\pi\pi$

fixing subtraction constants

- matching to large-N_c ChPT or Resonance Chiral Theory (are there any low-energy theorems $\sim$ Adler zero?)
  Beisert, Borasoy 2002; Escribano, Masjuan, Sant-Cillero 2010
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$\pi\eta$ phase shifts

- $\pi\eta$ scattering not well-known
- inverse amplitude method, other non-perturbative approaches?
  Oller, Oset, Peláez 1998; Gómez Nicola, Peláez 2002
- can we learn something about $\pi\eta$ phase-shifts?

Summary

Rigorous methods for ChPT-motivated investigations of hadronic meson decays:

- NREFT for perturbative final-state interactions in $\eta \to 3\pi$
  - understand the $\eta \to 3\pi^0$ slope parameter α:
    $\alpha_{\text{theo}} = -0.025 \pm 0.005$ vs. $\alpha_{\text{exp}} = -0.0317 \pm 0.0016$
  - rescattering ($\simeq$ imaginary parts) relate charged and neutral Dalitz parameters $\Rightarrow$ tension in experimental data (KLOE)

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  - ▷ rescattering ($\simeq$ imaginary parts) relate charged and neutral Dalitz parameters $\Rightarrow$ tension in experimental data (KLOE)

- non-perturbative final-state interactions via dispersion relations:
  - ▷ iterative numerical solution
  - ▷ input full $\pi\pi$ phase shifts
  - ▷ more appropriate for precision determination of quark mass ratios from $\eta \to 3\pi$
  - ▷ to be extended to hadronic η' decays

Spares

Non-relativistic EFT (1): basics

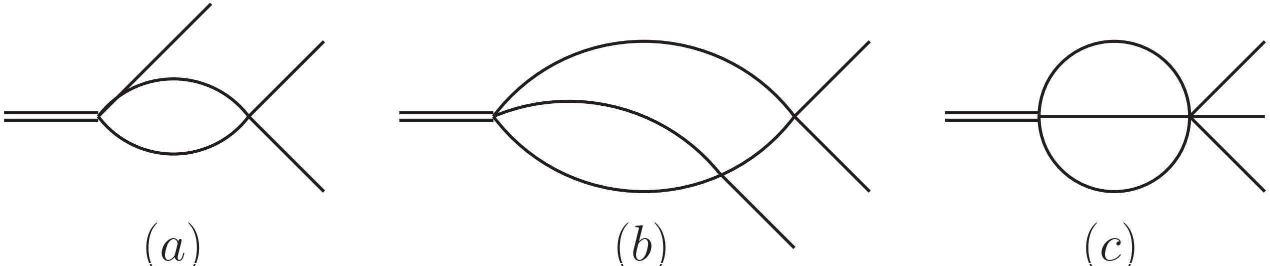
momenta	:	$\lvert\mathbf{p}\rvert/M_\pi = \mathcal{O}(\epsilon)$
kinetic energy	:	$T = \omega(\mathbf{p}) - M_\pi = \mathcal{O}(\epsilon^2)$
in $\eta \to 3\pi$	:	$M_\eta - \sum_i M_i = \sum_i T_i = \mathcal{O}(\epsilon^2)$

where $\omega(\mathbf{p}) = \sqrt{M_\pi^2 + \mathbf{p}^2}$

- non-relativistic region = whole decay region (and slightly beyond)
- two-fold expansion in ϵ and $\pi\pi$ scattering length a
- at given order a, ϵ, only finite number of graphs contribute
  $\Rightarrow$ power counting

Non-relativistic EFT (2): power counting

- organise tree level polynomials in even powers of momenta
  $\Rightarrow \mathcal{O}(\epsilon^0)$, $\mathcal{O}(\epsilon^2) \hat{=} a$, $\mathcal{O}(\epsilon^4) \hat{=} b, d, \alpha, \ldots$
- loops:



propagator: $$\frac{1}{\omega(\mathbf{p}) - p^0} = \mathcal{O}(\epsilon^{-2})$$

loop integration: $$d^4p = dp^0 d^3\mathbf{p} = \mathcal{O}(\epsilon^5)$$

- each loop with two-body rescattering $(\epsilon^{-2})^2\epsilon^5 = \mathcal{O}(\epsilon)$ suppressed
  (a) $= \mathcal{O}(a^1\epsilon^1)$
  (b) $= \mathcal{O}(a^2\epsilon^2)$ $\Rightarrow$ correlated expansion in a and ϵ
- loop with three-body rescattering $(\epsilon^{-2})^3(\epsilon^5)^2 = \mathcal{O}(\epsilon^4)$ suppressed
  (c) $= \mathcal{O}(\epsilon^4)$

Non-relativistic EFT (3): Lagrangian

- propagator: $$\underbrace{\frac{1}{M_\pi^2 - p^2}}_{\text{relativistic}} = \underbrace{\frac{1}{2\omega(\mathbf{p})}\frac{1}{\omega(\mathbf{p}) - p^0}}_{\text{"non-relativistic"}} + \underbrace{\frac{1}{2\omega(\mathbf{p})}\frac{1}{\omega(\mathbf{p}) + p^0}}_{\text{antiparticles}}$$

  generated by Lagrangian

  $$\mathcal{L}_{\text{kin}} = \Phi^\dagger(2W)(i\partial_t - W)\Phi \,, \qquad W = \sqrt{M_\pi^2 - \triangle}$$

  Note: non-local $\mathcal{L}_{\text{kin}}$ generates all relativistic corrections; manifestly Lorentz-invariant / frame-independent

- correctly reproduces singularity structure at small momenta $|\mathbf{p}| \ll M_\pi$, subsumes far-away singularities in effective couplings
- interaction terms:

$$\mathcal{L}_{\pi\pi} = C_x\big(\pi_-^\dagger \pi_+^\dagger (\pi_0)^2 + h.c.\big) + \big(\text{derivative terms}\big)$$

$$\mathcal{L}_{\eta 3\pi} = \frac{K_0}{6}\big(\eta^\dagger \pi_0^3 + h.c.\big) + L_0\big(\eta^\dagger \pi_0\pi_+\pi_- + h.c.\big) + \ldots$$

- Lagrangian-based QFT, analyticity + unitarity obeyed

Non-relativistic EFT (4): matching

- match the $\pi\pi$ coupling constants to the effective range expansion of the $\pi\pi$ scattering amplitude:

$$\mathrm{Re}\, T(\pi^+\pi^- \to \pi^0\pi^0) = 2C_x + \mathcal{O}(\epsilon^2)$$

$$2C_x = -\frac{32\pi}{3}(a_0 - a_2)\Big\{1 + \underbrace{\frac{M_{\pi^+}^2 - M_{\pi^0}^2}{3M_\pi^2}}_{\text{ChPT } \mathcal{O}(e^2)} + \ldots\Big\}$$

$$= -\frac{32\pi}{3}\Big\{a_0 - a_2 + \underbrace{(0.61 \pm 0.16) \times 10^{-2}}_{\text{ChPT } \mathcal{O}(e^2p^2)}\Big\}$$

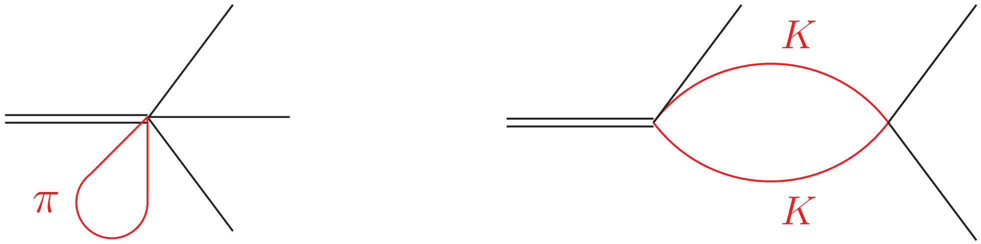
Knecht, Urech 1997; Gasser et al. 2001

isospin-breaking corrections in matching calculated in ChPT

- parametrise polynomial $\eta \to \pi^+\pi^-\pi^0$ in terms of $L_0, L_1, \ldots$
  $\eta \to 3\pi^0$ in terms of $K_0, K_1, \ldots$
- match decay parameters ($K_0, \ldots, L_0, \ldots$) to ChPT at $\mathcal{O}(p^4)$
  $\Leftrightarrow$ tree Dalitz plot parameters a^{tree}, b^{tree}, α^{tree} …

Comparison (1): NREFT vs. ChPT

– not predictive: a^{tree}, b^{tree}, α^{tree} ... input parameters, subsume e.g.



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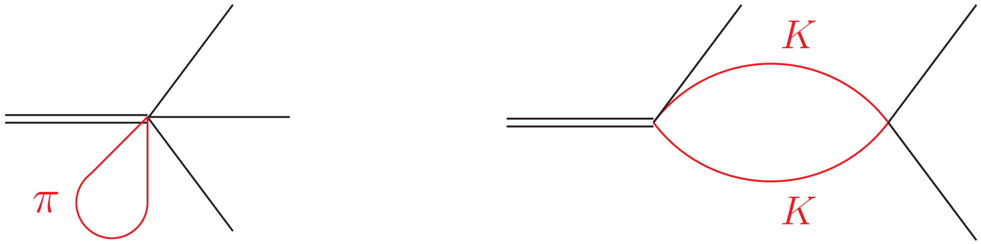

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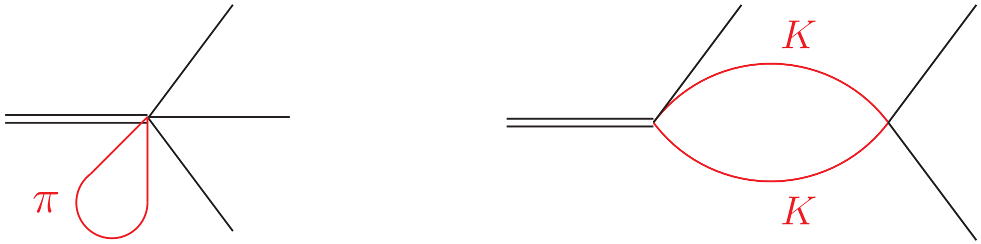

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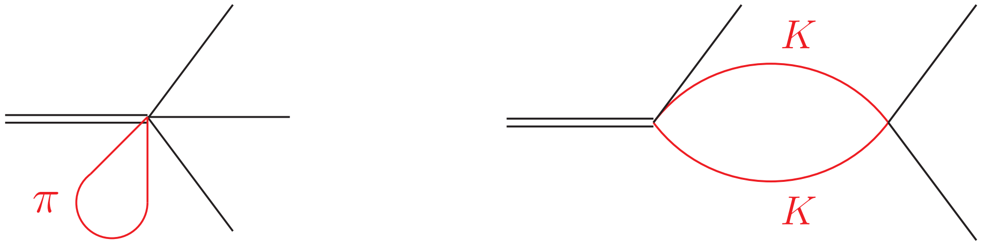

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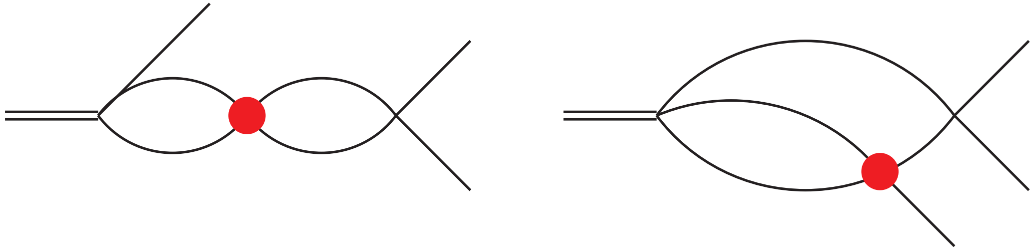

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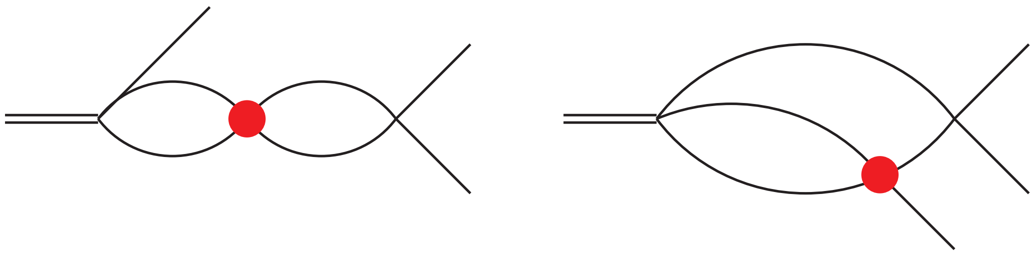

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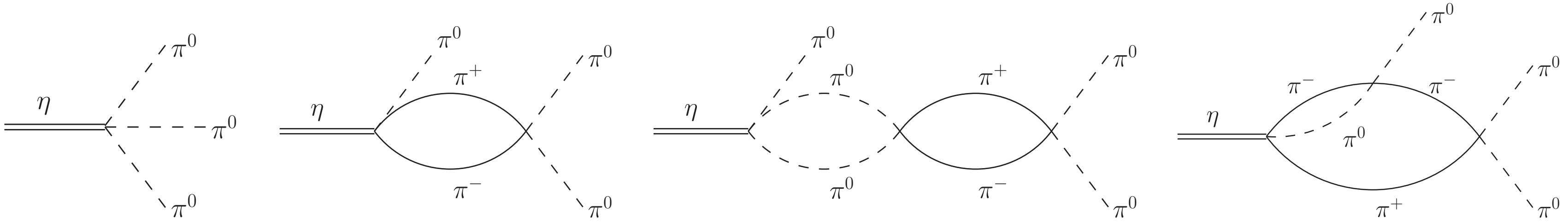
$+$ isospin breaking, kinematic effects due to $M_{\pi^+} \neq M_{\pi^0}$,
correct thresholds everywhere

Representation of $\eta \longrightarrow 3\pi$ amplitude up to two loops

- complete representation to $\mathcal{O}(\epsilon^4, a\epsilon^5, a^2\epsilon^4)$, partial $\mathcal{O}(a^2\epsilon^6, a^2\epsilon^8)$:



- only one-loop function purely imaginary (above $\pi\pi$ threshold):

$$J(s) = \frac{i\,v(s)}{16\pi}\,, \quad v(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

- non-trivial two-loop function purely real (above $\pi\pi$ threshold):

$$F(s) = \frac{v(s)}{256\pi^2}\sqrt{\frac{M_\eta^2 - 9M_\pi^2}{M_\eta^2 - M_\pi^2}} + \mathcal{O}\big(v(s)^3\big)$$

analytical representation in terms of arctan functions available

Bissegger et al. 2008

same goes for "bubble-sum" two-loop graph: $[J(s)]^2$ real

Understanding α in NREFT: power counting

1-loop vs. 2-loop

- power counting in $a_{\pi\pi}$:
  symbolically (in the center of the Dalitz plot):

$$\mathcal{A} = \mathcal{A}_{\text{tree}} + i\,\mathcal{A}_{1-\text{loop}}\,a_{\pi\pi} + \mathcal{A}_{2-\text{loop}}\,a_{\pi\pi}^2 + \mathcal{O}(i\,a_{\pi\pi}^3)$$

$$\Rightarrow \quad |\mathcal{A}|^2 = \mathcal{A}_{\text{tree}}^2 + \big(\mathcal{A}_{1-\text{loop}}^2 + \mathcal{A}_{\text{tree}} \times \mathcal{A}_{2-\text{loop}}\big)\,a_{\pi\pi}^2 + \mathcal{O}(a_{\pi\pi}^4)$$

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importance of rescattering

- power counting in ϵ: $\alpha^{\text{tree}} = \mathcal{O}(\epsilon^4)$
- loops: $\mathcal{A}_{1-\text{loop}} = \mathcal{O}(\epsilon)$, $\mathcal{A}_{2-\text{loop}} = \mathcal{O}(\epsilon^2)$
  contribute to all Dalitz plot parameters at $\mathcal{O}(a_{\pi\pi}^2\epsilon^2)$
  $\Rightarrow$ heightened importance of loops for higher slope parameters!

Charged Dalitz plot parameters

	a	b	d
KLOE 2008	$-1.090^{+0.009}_{-0.020}$	0.124 ± 0.012	$0.057^{+0.009}_{-0.017}$
Crystal Barrel 1998	-1.22 ± 0.07	0.22 ± 0.11	0.06 ± 0.04
ChPT $\mathcal{O}(p^4)$	-1.34 ± 0.04	0.43 ± 0.02	0.077 ± 0.008
ChPT $\mathcal{O}(p^6)$*	-1.27 ± 0.08	0.39 ± 0.10	0.055 ± 0.057
dispersive**	-1.16	$0.24\ldots0.26$	$0.09\ldots0.10$
$\mathcal{O}(p^4)$ + NREFT	-1.21 ± 0.02	0.31 ± 0.02	0.050 ± 0.002

*Bijnens, Ghorbani 2007

**Kambor, Wiesendanger, Wyler 1995

- note significant discrepancy theory vs. experiment (KLOE) for b
  large violation of current-algebra relation $b = a^2/4$
  about to be remeasured WASA-at-COSY