FOUR-BODY EFIMOV EFFECT

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OUTLINE OF THE TALK

- Introduction and physical motivation
- General theory for Efimovian N-mers
- The $3 + 1$ fermionic problem
- Comparison to existing predictions

INTRODUCTION

TWO-BODY PROBLEM

- For long-range interaction, infinite number of bound states may exist, with accumulation point at $E = 0$. Cf. hydrogen atom:

$$E_n \propto -\frac{1}{n^2}, \qquad n \in \mathbb{N}^*$$

- For short-range interaction, finite number of bound states. Cf. van der Waals interaction between two atoms.

- Can the situation change for more than two atoms ?

In all what follows, s-wave short-range interaction among ultracold atoms with infinite scattering length $|a| = \infty$.

THREE-BODY PROBLEM

- Efimov (1971): Three bosons, $1/a = 0$, no dimer state. Then there exists an infinite number of trimer states, $E = 0$ accumulation point, geometric spectrum:

$$E_n^{(3)} \underset{n \to +\infty}{\sim} E_{\text{ref}}^{(3)} e^{-2\pi n/|s_3|}$$

  where purely imaginary $s_3 = i \times 1.00624$ solves transcendental equation, $E_{\text{ref}}^{(3)}$ depends on microscopic details.

- Efimov (1973): Solution for three arbitrary particles, $1/a = 0$. E.g. Efimov trimers for two fermions (masse M, same spin state) and one impurity (masse m) if (Petrov, 2003)

$$\alpha \equiv \frac{M}{m} > \alpha_c(2;1) \simeq 13.607$$

  with $s_3(\alpha) \in i\mathbb{R}^{+*}$ from known transcendental equation.

ARE THERE EFIMOVIAN TETRAMERS ?

$$E_n^{(4)} \underset{n\to+\infty}{\sim} E_{\mathrm{ref}}^{(4)} e^{-2\pi n/|s_4|} \ ?$$

Negative results:

- **Amado, Greenwood (1973): "There is No Efimov effect for Four or More Particles". Explanation: Case of bosons, there exist trimers, tetramers decay.**
- **Hammer, Platter (2007), von Stecher, D'Incao, Greene (2009), Deltuva (2010): The four-boson problem (here $1/a = 0$) depends only on $E_{\mathrm{ref}}^{(3)}$, no $E_{\mathrm{ref}}^{(4)}$ to add.**
- **Key point: $N = 3$ Efimov effect breaks separability in hyperspherical coordinates for $N = 4$.**

Idea: Consider three fermions (M) and one impurity (m).

GENERAL THEORY FOR EFIMOVIAN N-MERS $(N \geq 3)$

THE ZERO-RANGE WIGNER-BETHE-PEIERLS MODEL

- Interactions are replaced by contact conditions on the wavefunction.
- For $r_{ij} \to 0$ with fixed ij-centroid $\vec{C}_{ij} = (m_i\vec{r}_i + m_j\vec{r}_j)/(m_i + m_j)$ different from $\vec{r}_k, k \neq i, j$:

$$\psi(\vec{r}_1, \ldots, \vec{r}_N) = \left(\frac{1}{r_{ij}} - \frac{1}{a}\right) A_{ij}[\vec{C}_{ij}; (\vec{r}_k)_{k\neq i,j}] + O(r_{ij})$$

- Elsewhere, non interacting Schrödinger equation

$$E\psi = \sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\Delta_{\vec{r}_i}\psi$$

  with correct exchange symmetry
- Scale invariance: $\psi_\lambda(\vec{r}_1, \ldots, \vec{r}_N) \equiv \psi(\vec{r}_1/\lambda, \ldots, \vec{r}_N/\lambda)$ is another solution with eigenenergy E/λ^2.

SEPARABILITY IN HYPERSPHERICAL COORDINATES

Werner, Castin (2006):

- Use Jacobi coordinates to separate center of mass $\vec{C}$
- Hyperspherical coordinates: $(\vec{r}_1, \ldots, \vec{r}_N) \leftrightarrow (\vec{C}, R, \vec{\Omega})$ with $3N-4$ hyperangles $\vec{\Omega}$ and the hyperradius such that $m_u R^2 = \sum_{i=1}^{N} m_i (\vec{r}_i - \vec{C})^2$
- Hamiltonian is clearly separable:

$$H = -\frac{\hbar^2}{2m_u}\left[\partial_R^2 + \frac{3N-4}{R}\partial_R + \frac{1}{R^2}\Delta_{\vec{\Omega}}\right]$$

- Do the contact conditions preserve separability ? Yes for $E = 0$, due to scaling invariance: $\psi_0 = R^\nu \phi(\vec{\Omega})$. Solves $\Delta_{\vec{\Omega}}\phi(\vec{\Omega}) = -\left[s_N^2 - \left(\frac{3N-5}{2}\right)^2\right]\phi(\vec{\Omega})$ with contact conditions, $s_N^2 \in \mathbb{R}$ belongs to a discrete set, and $\nu = s_N - \frac{3N-5}{2}$.

- For arbitrary energy, Ansatz with $E = 0$ hyperrangular part
$$\psi = F(R) R^{-(3N-5)/2} \phi(\vec{\Omega})$$
obeys contact conditions $[R^2 = R^2(r_{ij} = 0) + O(r_{ij}^2)]$.
- Schrödinger equation for a fictitious particle in 2D:
$$EF(R) = -\frac{\hbar^2}{2m_u}\left[F''(R) + \frac{1}{R}F'(R)\right] + \frac{\hbar^2 s_N^2}{2m_u R^2}$$
- There exist Efimovian N-meres $\Longleftrightarrow$ there exists $s_N^2 < 0$
- Fall to the center, H not self-adjoint. Impose N-body contact condition with new parameter q:
$$F(R) \underset{R \to 0}{\sim} (qR)^{s_N} + (qR)^{-s_N}$$
Discrete scaling invariance $\lambda^{2s_N} = 1$, geometric spectrum.

CRUCIAL POINTS OF GENERAL THEORY

- To find N-body Efimov effect, one simply needs to calculate the exponents s_N, that is to solve the Wigner-Bethe-Peierls model at zero energy:

$$\psi_0(\vec{r}_1, \ldots, \vec{r}_N) = R^{s_N-(3N-5)/2}\phi(\vec{\Omega})$$

- General theory OK if $\Delta_{\vec{\Omega}}$ self-adjoint: no n-body Efimov effect $\forall n \le N-1$.

THE $3+1$ FERMIONIC PROBLEM

(Castin, Mora, Pricoupenko, 2010)

INTEGRAL EQUATION

- Three fermions (mass M, same spin state) and one impurity (mass m)
- General theory OK for a mass ratio

$$\alpha \equiv \frac{M}{m} < \alpha_c(2;1) \simeq 13.607$$

- Calculate $E = 0$ solution in momentum space. An integral equation for Fourier transform of A_{ij}:

$$0 = \left[\frac{1+2\alpha}{(1+\alpha)^2}(k_1^2 + k_2^2) + \frac{2\alpha}{(1+\alpha)^2}\vec{k}_1 \cdot \vec{k}_2\right]^{1/2} D(\vec{k}_1, \vec{k}_2)$$

$$+ \int \frac{d^3k_3}{2\pi^2} \frac{D(\vec{k}_1, \vec{k}_3) + D(\vec{k}_3, \vec{k}_2)}{k_1^2 + k_2^2 + k_3^2 + \frac{2\alpha}{1+\alpha}(\vec{k}_1 \cdot \vec{k}_2 + \vec{k}_1 \cdot \vec{k}_3 + \vec{k}_2 \cdot \vec{k}_3)}$$

- D has to obey fermionic symmetry.

REDUCTION OF THE INTEGRAL EQUATION

Rotational invariance:

- D is the $m_l = 0$ component of a spinor of spin l:

$$\vec{D}(\vec{k}_1, \vec{k}_2) = {}^t\rho\, \vec{D}(\mathcal{R}\vec{k}_1, \mathcal{R}\vec{k}_2)$$

- Clever choice of the rotation matrix $\mathcal{R}$:

$$\vec{D}(\vec{k}_1, \vec{k}_2) = {}^t\rho \quad \underbrace{\vec{D}[k_1\vec{e}_x, k_2(\cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y)]}_{2l+1 \text{ unknown functions } f^{(l)}_{m_l}(k_1,k_2,\theta)}$$

Scaling invariance for $E = 0$:

$$f^{(l)}_{m_l}(k_1, k_2, \theta) = (k_1^2 + k_2^2)^{-(s_4+7/2)/2}(\cosh x)^{3/2}\Phi^{(l)}_{m_l}(x, \theta)$$

with $x = \ln(k_2/k_1)$.

The integral equation gives $M^{(l)}_{s_4}[\vec{\Phi}^{(l)}] = 0$.

s_4 allowed $\Longleftrightarrow$ $M^{(l)}_{s_4}$ has a zero eigenvalue

RESULTS

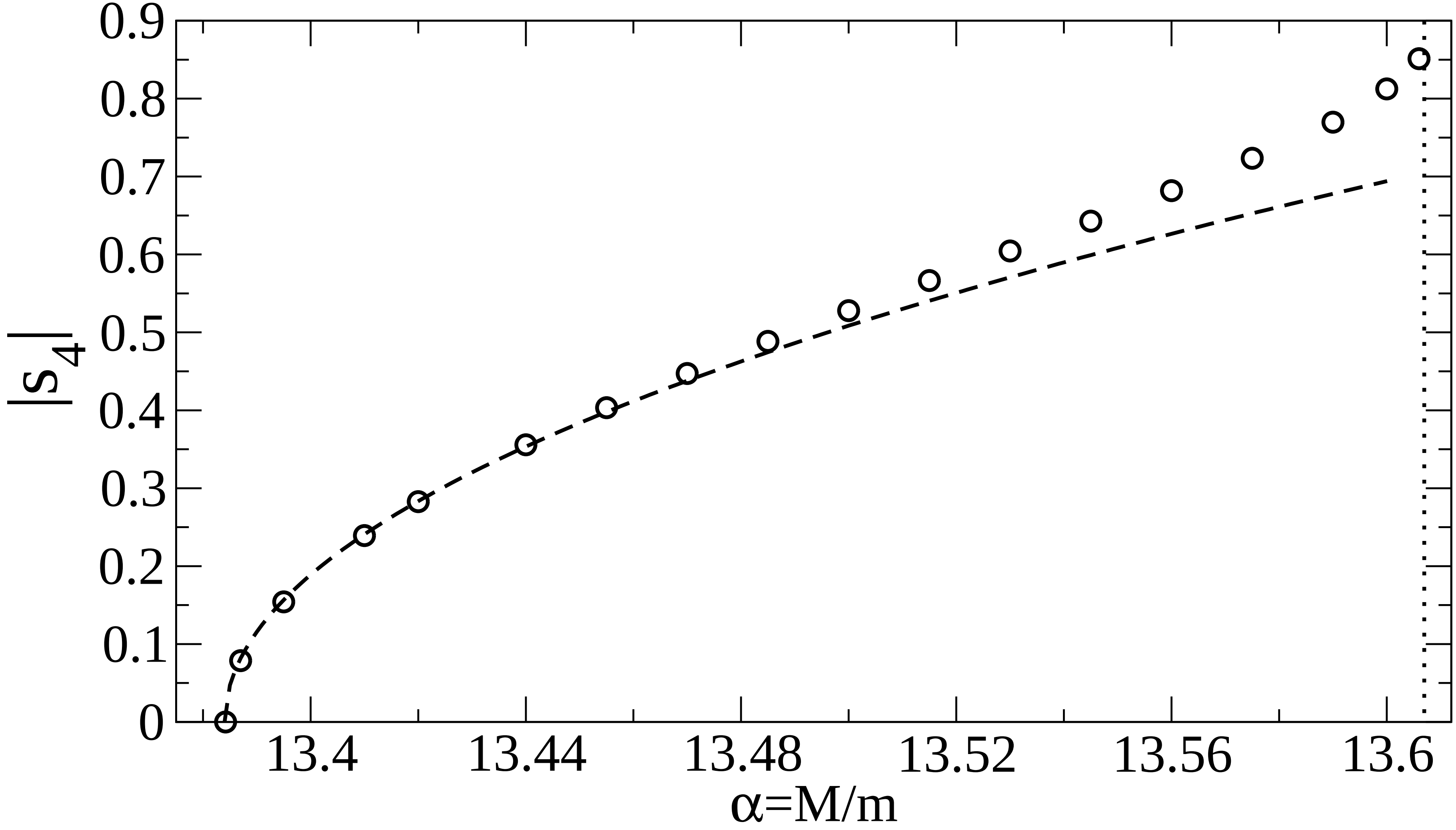
- Numerical exploration up to $l = 10$
- Four-body Efimov effect obtained for a single s_4, in channel $l = 1$ with even parity:

$$D(\vec{k}_1, \vec{k}_2) = \vec{e}_z \cdot \frac{\vec{k}_1 \times \vec{k}_2}{||\vec{k}_1 \times \vec{k}_2||} \, f_0^{(1)}(k_1, k_2, \theta)$$

in the interval of mass ratio

$$\alpha_c(3;1) \simeq 13.384 < \alpha < \alpha_c(2;1) \simeq 13.607$$

NUMERICAL VALUES OF $s_4 \in i\mathbb{R}$



EXPERIMENTAL ASPECTS

- Large scattering length with magnetic Feshbach resonance (Grimm, 2006; Hulet, 2009)
- Radio-frequency spectroscopy of trimers (Jochim, 2010)
- Remaining issue: Narrow interval of mass ratio.

Solution 1: The right mixture

- ^{41}Ca and ^{3}He* have mass ratio $\alpha \simeq 13.58 \in [13.384, 13.607]$
- A priori, $|s_4| \simeq 0.75$ large enough to see two tetramer states
- ^{41}Ca has same radioactivity as ^{239}Pu (half-life 10^5 years)

Solution 2: Mass tuning

- ^{40}K and ^{3}He* have slightly-off mass ratio $\alpha \simeq 13.25$
- Use optical lattice to tune effective mass (Petrov, Shlyapnikov, 2007)

COMPARISON TO PREVIOUS WORKS

MINLOS'S THEOREM (1995)

Theorem: *In the $n+1$ fermionic problem, the Wigner-Bethe-Peierls Hamiltonian is bounded from below if and only if*

$$(n-1)\frac{2\alpha(1+1/\alpha)^3}{\pi\sqrt{1+2\alpha}}\int_0^{\mathrm{asin}\frac{\alpha}{1+\alpha}} dt\, t\sin t < 1.$$

- We expect that "not bounded from below" is equivalent to "with Efimov effect".
- Case $n=3$: $\alpha_c^{\mathrm{Minlos}} \simeq 5.29$ totally differs from ours...
- Case $\alpha=1$: No stable unitary gas for $n>9$...
- Weak point: Proof not included in Minlos' paper.
- Recent proof: Teta, Finco (2010). But we have found a hole in the proof.